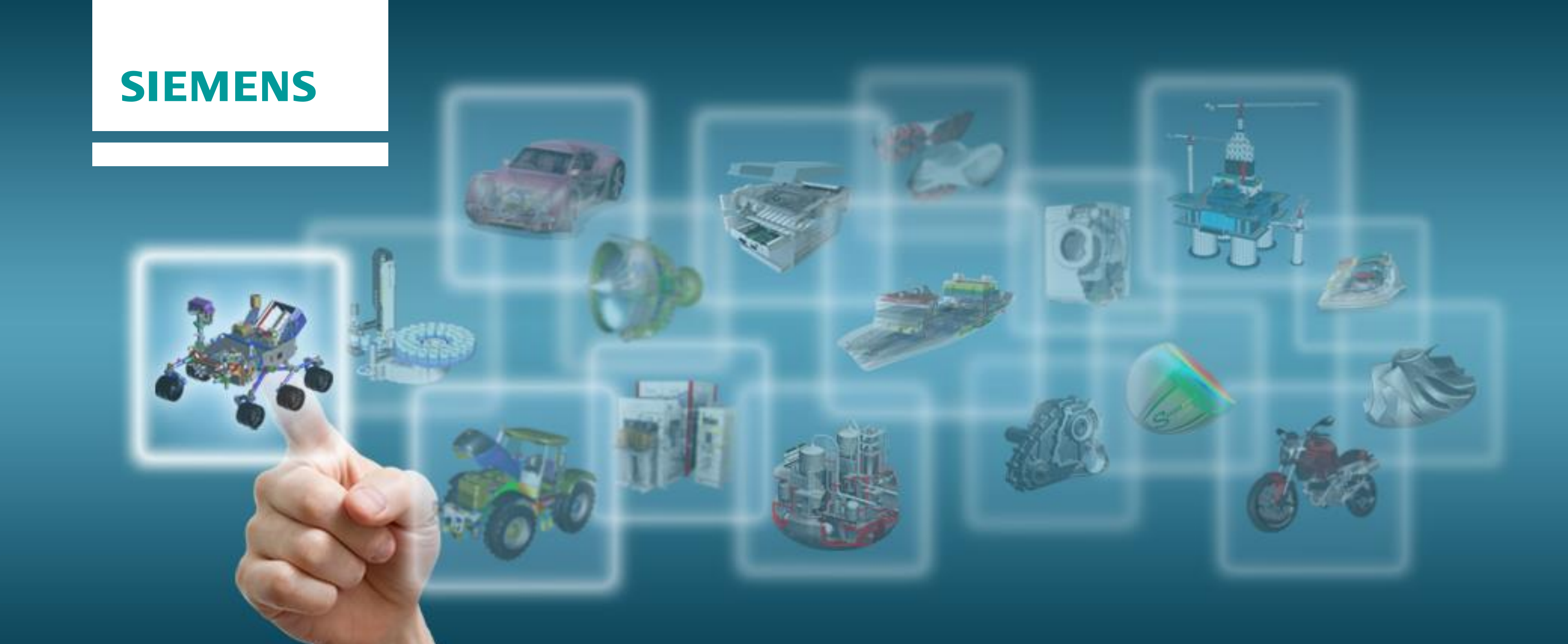

ARRAYCON - ESR1

Eigenfunctions for clamped rectangular orthotropic plates

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Smarter decisions, better products.

Notation

EQUATION OF MOTION FOR ORTHOTROPIC RECTANGULAR PLATE

$$D_1 \frac{\partial^4 w(x,y,t)}{\partial x^4} + 2D_3 \frac{\partial^4 w(x,y,t)}{\partial x^2 \partial y^2} + D_2 \frac{\partial^4 w(x,y,t)}{\partial y^4} + \rho h \frac{\partial^2 w(x,y,t)}{\partial t^2} = 0$$

$$D_1 = \frac{E_1 h^3}{12(1-\vartheta_{12}\vartheta_{21})},\ D_2 = \frac{E_2 h^3}{12(1-\vartheta_{12}\vartheta_{21})},\ D_{66} = \frac{G_{12} h^3}{12},\ D_3 = D_{12} + 2D_{66},\ D_{12} = \vartheta_{12} D_2 = \vartheta_{21} D_1$$

If $E_1 = E_2 = E$ & $\vartheta_{12} = \vartheta_{21} = \vartheta$,

$$D_1 = D_2 = \frac{Eh^3}{12(1-\vartheta^2)} = D \ \&\ D_3 = D_{12} + 2D_{66} = \frac{\vartheta E h^3}{12(1-\vartheta^2)} + 2\frac{Eh^3}{12*2*(1-\vartheta)} = \frac{Eh^3}{12(1-\vartheta^2)} = D$$

Then Isotropic Homogenouse case

$$D\Delta\Delta w(x,y,t) + \rho h \frac{\partial^2 w(x,y,t)}{\partial t^2} = 0$$

Eigenfunction 2

SEPARATION OF VARIABLES

$$w(x,y,t) = \Phi(x)\psi(y)\big(A\cos(\omega t) + B\sin(\omega t)\big)$$

Then

$$D_1 \frac{\partial^4 \Phi(x)}{\partial x^4}\psi(y) + 2D_3 \frac{\partial^4 \Phi(x)\psi(y)}{\partial x^2 \partial y^2} + D_2 \frac{\partial^4 \psi(y)}{\partial y^4}\Phi(x) - \beta^4 \Phi(x)\psi(y) = 0$$

Which $\beta^4 = \omega^2 \rho h$

In order for the separation occur,

$$\frac{\partial^2 \Phi(x)}{\partial x^2} = -\mu^2 \Phi(x) \ \& \ \frac{\partial^2 \psi(y)}{\partial y^2} = -\lambda^2 \psi(y)$$

$$\Rightarrow \ \Phi(x) = Ae^{\mu x} \ \& \ \psi(y) = Be^{\lambda y}$$

Then

$$D_1 \mu^4 + 2D_3 \mu^2 \lambda^2 + D_2 \lambda^4 - \beta^4 = 0$$

Eigenfunction 2

Roots

$$\mu_{1,2} = \pm i\sqrt[2]{\varepsilon_1 + \delta_1} \equiv \pm i\alpha_1 \;\&\; \mu_{3,4} = \pm\sqrt[2]{\varepsilon_1 - \delta_1} \equiv \pm\beta_1$$

$$\varepsilon_1 = \sqrt{\lambda^4\left[\left(\frac{D_3}{D_1}\right)^2 - \frac{D_1}{D_2}\right] + \frac{\beta^4}{D_1}} \;\&\; \delta_1 = \lambda^2\frac{D_3}{D_1}$$

$$\lambda_{1,2} = \pm i\sqrt[2]{\varepsilon_2 + \delta_2} \equiv \pm i\alpha_2 \;\&\; \lambda_{3,4} = \pm\sqrt[2]{\varepsilon_2 - \delta_2} \equiv \pm\beta_2$$

$$\varepsilon_2 = \sqrt{\mu^4\left[\left(\frac{D_3}{D_2}\right)^2 - \frac{D_2}{D_1}\right] + \frac{\beta^4}{D_2}} \;\&\; \delta_2 = \mu^2\frac{D_3}{D_2}$$

At the same time

$$\alpha_2 = \sqrt{\sqrt{{\alpha_1}^4\left[\left(\frac{D_3}{D_2}\right)^2 - \frac{D_2}{D_1}\right] + \frac{\beta^4}{D_2}} - {\alpha_1}^2\frac{D_3}{D_2}} \;\&\; \beta_2 = \sqrt{\sqrt{{\alpha_1}^4\left[\left(\frac{D_3}{D_2}\right)^2 - \frac{D_2}{D_1}\right] + \frac{\beta^4}{D_2}} + {\alpha_1}^2\frac{D_3}{D_2}}$$

$$\left({\alpha_1}^2 + {\beta_1}^2\right)^2 + \frac{D_1 D_2 - {D_3}^2}{{D_3}^2}\left({\alpha_1}^2 - {\beta_1}^2\right)^2 = \frac{4\beta^4}{D_1}$$

Eigenfunction 2

SOLUTION

$$\Phi(x) = A_1 cos\alpha_1 x + B_1 sin\alpha_1 x + C_1 cosh\beta_1 x + H_1 sinh\beta_1 x$$
$$\psi(y) = A_2 cos\alpha_2 y + B_2 sin\alpha_2 y + C_2 cosh\beta_2 y + H_2 sinh\beta_2 y$$

Boundary conditions:

In x direction:

$$w(0,y) = 0 \Rightarrow \Phi(0) = 0$$
$$w(a,0) = 0 \Rightarrow \Phi(a) = 0$$
$$\frac{\partial w(0,y)}{\partial x} = 0 \Rightarrow \frac{\partial \Phi(0)}{\partial x} = 0$$
$$\frac{\partial w(a,y)}{\partial x} = 0 \Rightarrow \frac{\partial \Phi(a)}{\partial x} = 0$$

The same in y direction

$$A_1 = -C_1$$
$$\alpha_1 B_1 = -\beta_1 H_1$$

with $k_2 = \frac{cos\alpha_1 a - cosh\beta_1 a}{(\beta_1/\alpha_1) sin\alpha_1 a - sinh\beta_1 a}$

$$H_1 = -k_2 C_1$$

Eigenfunction 2

Normal eigenfunctions

Assuming $C_1 = 1$

$$\Phi(x) = -cos\alpha_1 x + (\beta_1/\alpha_1)k_2 sin\alpha_1 x + cosh\beta_1 x - k_2 sinh\beta_1 x$$

The same for y direciton

$$\psi(y) = -cos\alpha_2 y + (\beta_2/\alpha_2)k_1 sin\alpha_2 y + cosh\beta_2 y - k_1 sinh\beta_2 y$$

Orthogonality?!

Use the same re-orthogonalisation process of eigenfunction 1

Eigenfunction 2

PROOF OF NORMAL EIGENFUNCTIONS

$$w(x,y,t) = \Phi(x)\psi(y)$$

Then for the eigenfunction:

$$D_1 \frac{\partial^4 \Phi(x)}{\partial x^4}\psi(y) + 2D_3 \frac{\partial^4 \Phi(x)\psi(y)}{\partial x^2 \partial y^2} + D_2 \frac{\partial^4 \psi(y)}{\partial y^4}\Phi(x) - \delta \cdot \Phi(x)\psi(y) = 0$$

In order for the separation occur,

$$\frac{\partial^2 \Phi(x)}{\partial x^2} = -\mu^2 \Phi(x) \; \& \; \frac{\partial^2 \psi(y)}{\partial y^2} = -\lambda^2 \psi(y)$$

$$\Rightarrow \; \Phi(x) = Ae^{\mu x} \; \& \; \psi(y) = Be^{\lambda y}$$

Then

$$D_1 \mu^4 + 2D_3 \mu^2 \lambda^2 + D_2 \lambda^4 - \delta = 0$$

Eigenfunction 2

PROOF OF NORMAL EIGENFUNCTIONS

$$\mu_{1,2} = \pm i\sqrt[2]{\varepsilon_1 + \delta_1} \equiv \pm i\alpha_1 \ \& \ \mu_{3,4} = \pm\sqrt[2]{\varepsilon_1 - \delta_1} \equiv \pm\beta_1$$

$$\varepsilon_1 = \sqrt{\lambda^4\left[\left(\frac{D_3}{D_1}\right)^2 - \frac{D_1}{D_2}\right] + \frac{\delta}{D_1}} \ \& \ \delta_1 = \lambda^2\frac{D_3}{D_1}$$

$$\lambda_{1,2} = \pm i\sqrt[2]{\varepsilon_2 + \delta_2} \equiv \pm i\alpha_2 \ \& \ \lambda_{3,4} = \pm\sqrt[2]{\varepsilon_2 - \delta_2} \equiv \pm\beta_2$$

$$\varepsilon_2 = \sqrt{\mu^4\left[\left(\frac{D_3}{D_2}\right)^2 - \frac{D_2}{D_1}\right] + \frac{\delta}{D_2}} \ \& \ \delta_2 = \mu^2\frac{D_3}{D_2}$$

At the same time

$$\alpha_2 = \sqrt{\sqrt{{\alpha_1}^4\left[\left(\frac{D_3}{D_2}\right)^2 - \frac{D_2}{D_1}\right] + \frac{\delta}{D_2}} - {\alpha_1}^2\frac{D_3}{D_2}} \ \& \ \beta_2 = \sqrt{\sqrt{{\alpha_1}^4\left[\left(\frac{D_3}{D_2}\right)^2 - \frac{D_2}{D_1}\right] + \frac{\delta}{D_2}} + {\alpha_1}^2\frac{D_3}{D_2}}$$

$$\left({\alpha_1}^2 + {\beta_1}^2\right)^2 + \frac{D_1 D_2 - {D_3}^2}{{D_3}^2}\left({\alpha_1}^2 - {\beta_1}^2\right)^2 = \frac{4\delta}{D_1}$$

Eigenfunction 2

PROOF OF NORMAL EIGENFUNCTIONS

$$\Phi(x) = A_1 cos\alpha_1 x + B_1 sin\alpha_1 x + C_1 cosh\beta_1 x + H_1 sinh\beta_1 x$$
$$\psi(y) = A_2 cos\alpha_2 y + B_2 sin\alpha_2 y + C_2 cosh\beta_2 y + H_2 sinh\beta_2 y$$

Boundary conditions:

In x direction:

$$w(0,y) = 0 \Rightarrow \Phi(0) = 0$$
$$w(a,0) = 0 \Rightarrow \Phi(a) = 0$$
$$\frac{\partial w(0,y)}{\partial x} = 0 \Rightarrow \frac{\partial \Phi(0)}{\partial x} = 0$$
$$\frac{\partial w(a,y)}{\partial x} = 0 \Rightarrow \frac{\partial \Phi(a)}{\partial x} = 0$$

The same in y direction

$$A_1 = -C_1$$
$$\alpha_1 B_1 = -\beta_1 H_1$$

with $k_2 = \frac{cos\alpha_1 a - cosh\beta_1 a}{(\beta_1/\alpha_1) sin\alpha_1 a - sinh\beta_1 a}$

$$H_1 = -k_2 C_1$$

Eigenfunction 2

PROOF OF NORMAL EIGENFUNCTIONS

Assuming $C_1 = 1$

$$\Phi(x) = -cos\alpha_1 x + (\beta_1/\alpha_1)k_2 sin\alpha_1 x + cosh\beta_1 x - k_2 sinh\beta_1 x$$

The same for y direciton

$$\psi(y) = -cos\alpha_2 y + (\beta_2/\alpha_2)k_1 sin\alpha_2 y + cosh\beta_2 y - k_1 sinh\beta_2 y$$

Eigenfunction 2

Proof of normal eigenfunctions

$$\Phi_1 = A_1 cos\alpha_1 x + B_1 sin\alpha_1 x, \qquad \Phi_2 = C_1 cosh\beta_1 x + H_1 sinh\beta_1 x$$
$$\psi_1 = A_2 cos\alpha_2 y + B_2 sin\alpha_2 y, \qquad \psi_2 = C_2 cosh\beta_2 y + H_2 sinh\beta_2 y$$

$$\Phi(x) = \Phi_1 + \Phi_2, \qquad \psi(y) = \psi_1 + \psi_2$$

$$D_1 \frac{\partial^4 \Phi(x)}{\partial x^4}\psi(y) = D_1 \frac{\partial^4(\Phi_1+\Phi_2)}{\partial x^4}(\psi_1+\psi_2) = D_1\,({\alpha_1}^4\Phi_1 + {\beta_1}^4\Phi_2)(\psi_1+\psi_2)$$
$$2D_3\frac{\partial^4 w(x,y,t)}{\partial x^2 \partial y^2} = 2D_3\frac{\partial^4(\Phi_1+\Phi_2)(\psi_1+\psi_2)}{\partial x^2\partial y^2} = 2D_3({\alpha_1}^2{\alpha_2}^2\Phi_1\psi_1 - {\alpha_1}^2{\beta_2}^2\Phi_1\psi_2 - {\beta_1}^2{\alpha_2}^2\Phi_2\psi_1 + {\beta_1}^2{\beta_2}^2\Phi_2\psi_2)$$
$$D_2\frac{\partial^4\psi(y)}{\partial y^4}\Phi(x) = D_2\frac{\partial^4(\psi_1+\psi_2)}{\partial y^4}(\Phi_1+\Phi_2) = D_2(\Phi_1+\Phi_2)\left({\alpha_2}^4\psi_1 + {\beta_2}^4\psi_2\right)$$

$$\begin{aligned} D_1\frac{\partial^4\Phi(x)}{\partial x^4}\psi(y) + 2D_3\frac{\partial^4\Phi(x)\psi(y)}{\partial x^2\partial y^2} + D_2\frac{\partial^4\psi(y)}{\partial y^4}\Phi(x) \\ = (D_1{\alpha_1}^4 + 2D_3{\alpha_1}^2{\alpha_2}^2 + D_2{\alpha_2}^4)\Phi_1\psi_1 + \left(D_1{\alpha_1}^4 - 2D_3{\alpha_1}^2{\beta_2}^2 + D_2{\beta_2}^4\right)\Phi_1\psi_2 \\ + (D_1{\beta_1}^4 - 2D_3{\beta_1}^2{\alpha_2}^2 + D_2{\alpha_2}^4)\Phi_2\psi_1 + (D_1{\beta_1}^4 + 2D_3{\beta_1}^2{\beta_2}^2 + D_2{\beta_2}^4)\Phi_2\psi_2 \end{aligned}$$

Eigenfunction 2

PROOF OF NORMAL EIGENFUNCTIONS

$$\alpha_2 = \sqrt{\sqrt{\alpha_1{}^4\left[\left(\frac{D_3}{D_2}\right)^2 - \frac{D_2}{D_1}\right] + \frac{\delta}{D_2}} - \alpha_1{}^2\frac{D_3}{D_2}} \;\&\; \beta_2 = \sqrt{\sqrt{\alpha_1{}^4\left[\left(\frac{D_3}{D_2}\right)^2 - \frac{D_2}{D_1}\right] + \frac{\delta}{D_2}} + \alpha_1{}^2\frac{D_3}{D_2}}$$

$$\alpha_2 = \sqrt{\sqrt{\beta_1{}^4\left[\left(\frac{D_3}{D_2}\right)^2 - \frac{D_1}{D_2}\right] + \frac{\delta}{D_2}} - \beta_1{}^2\frac{D_3}{D_2}} \;\&\; \beta_2 = \sqrt{\sqrt{\beta_1{}^4\left[\left(\frac{D_3}{D_2}\right)^2 - \frac{D_1}{D_2}\right] + \frac{\delta}{D_2}} + \beta_1{}^2\frac{D_3}{D_2}}$$

$$(D_1\alpha_1{}^4 + 2D_3\alpha_1{}^2\alpha_2{}^2 + D_2\alpha_2{}^4) = \delta$$
$$\left(D_1\alpha_1{}^4 - 2D_3\alpha_1{}^2\beta_2{}^2 + D_2\beta_2{}^4\right) = \delta$$
$$\left(D_1\beta_1{}^4 - 2D_3\beta_1{}^2\alpha_2{}^2 + D_2\alpha_2{}^4\right) = \delta$$
$$\left(D_1\beta_1{}^4 + 2D_3\beta_1{}^2\beta_2{}^2 + D_2\beta_2{}^4\right) = \delta$$

Eigenfunction 2

PROOF OF NORMAL EIGENFUNCTIONS

$$D_1 \frac{\partial^4 \Phi(x)}{\partial x^4} \psi(y) + 2D_3 \frac{\partial^4 \Phi(x)\psi(y)}{\partial x^2 \partial y^2} + D_2 \frac{\partial^4 \psi(y)}{\partial y^4} \Phi(x) = \delta(\Phi_1\psi_1 + \Phi_1\psi_2 + \Phi_2\psi_1 + \Phi_2\psi_2)$$

$$D_1 \frac{\partial^4 \Phi(x)}{\partial x^4} \psi(y) + 2D_3 \frac{\partial^4 \Phi(x)\psi(y)}{\partial x^2 \partial y^2} + D_2 \frac{\partial^4 \psi(y)}{\partial y^4} \Phi(x) = \delta\Phi(x)\,\psi(y)$$

Which $\delta = \beta^4 = \omega^2 \rho h$

Eigenfunction 2

PROOF OF SELF-ADJOINT OPERATOR

$$L = D_1 \frac{\partial^4}{\partial x^4} + 2D_3 \frac{\partial^4}{\partial x^2 \partial y^2} + D_2 \frac{\partial^4}{\partial y^4}$$

Assuming w_1 & w_2 two function satisfy $Lw = \lambda w$ and the boundary conditions $\mathrm{w} = \frac{\partial w}{\partial x} = 0$ for x=0,a and $\mathrm{w} = \frac{\partial w}{\partial y} = 0$ for y=0,b

$$\langle Lu, v \rangle = \iint Lu \cdot v dxdy$$

$$\langle Lu, v \rangle = \langle u, Lv \rangle \,?$$

$$\int \int \frac{\partial^4 w_1}{\partial x^4} w_2 \, dxdy = \int \left[\frac{\partial^3 w_1}{\partial x^3} w_2 \bigg|_{x=0,a} - \int \left(\frac{\partial^3 w_1}{\partial x^3} \frac{\partial w_2}{\partial x} \right) dx \right] dy = -\int \left[\int \left(\frac{\partial^3 w_1}{\partial x^3} \frac{\partial w_2}{\partial x} \right) dx \right] dy$$

$$= -\int \left[\frac{\partial^2 w_1}{\partial x^2} \frac{\partial w_2}{\partial x} \bigg|_{x=0,a} - \int \left(\frac{\partial^2 w_1}{\partial x^2} \frac{\partial^2 w_2}{\partial x^2} \right) dx \right] dy = \int \left[\int \left(\frac{\partial^2 w_1}{\partial x^2} \frac{\partial^2 w_2}{\partial x^2} \right) dx \right] dy$$

$$= \int \left[\frac{\partial w_1}{\partial x} \frac{\partial^2 w_2}{\partial x^2} \bigg|_{x=0,a} - \int \left(\frac{\partial w_1}{\partial x} \frac{\partial^3 w_2}{\partial x^3} \right) dx \right] dy = -\int \left[\int \left(\frac{\partial w_1}{\partial x} \frac{\partial^3 w_2}{\partial x^3} \right) dx \right] dy = -\int \left[w_1 \frac{\partial^3 w_2}{\partial x^3} \bigg|_{x=0,a} - \int \left(w_1 \frac{\partial^4 w_2}{\partial x^4} \right) dx \right] dy$$

$$= \int \int w_1 \frac{\partial^4 w_2}{\partial x^4} dxdy$$

In the same process, $,\int \int \frac{\partial^4 w_1}{\partial y^4} w_2 \, dxdy = \int \int w_1 \frac{\partial^4 w_2}{\partial y^4} dxdy$

Eigenfunction 2

PROOF OF SELF-ADJOINT OPERATOR

$$\int\int \frac{\partial^4 w_1}{\partial x^2 \partial y^2} w_2 \, dxdy = \int \left[\frac{\partial^3 w_1}{\partial x \partial y^2} w_2 \Big|_{x=0,a} - \int \left(\frac{\partial^3 w_1}{\partial x \partial y^2} \frac{\partial w_2}{\partial x} \right) dx \right] dy = -\int \left[\int \left(\frac{\partial^3 w_1}{\partial x \partial y^2} \frac{\partial w_2}{\partial x} \right) dx \right] dy$$

$$= -\int \left[\frac{\partial^2 w_1}{\partial y^2} \frac{\partial w_2}{\partial x} \Big|_{x=0,a} - \int \left(\frac{\partial^2 w_1}{\partial y^2} \frac{\partial^2 w_2}{\partial x^2} \right) dx \right] dy = \int\int \frac{\partial^2 w_1}{\partial y^2} \frac{\partial^2 w_2}{\partial x^2} dxdy = \cdots = \int\int w_1 \frac{\partial^4 w_2}{\partial x^2 \partial y^2} dxdy$$

So

$$\langle Lw_1, w_2 \rangle = \langle w_1, L^* w_2 \rangle \ \ \& L = L^*$$

with $\langle Lu, v \rangle = \iint Lu \cdot v dxdy$

Then

$$\langle Lw_n, w_m \rangle - \langle w_n, Lw_m \rangle = 0$$

$$\langle \lambda_n - \lambda_m \rangle \cdot \langle w_n, w_m \rangle = 0$$

$$\lambda_n - \lambda_m \neq 0 \Rightarrow \langle w_n, w_m \rangle = 0$$

Notation

EQUATION OF MOTION FOR ORTHOTROPIC RECTANGULAR PLATE

$$D_1 \frac{\partial^4 w(x,y,t)}{\partial x^4} + 2D_3 \frac{\partial^4 w(x,y,t)}{\partial x^2 \partial y^2} + D_2 \frac{\partial^4 w(x,y,t)}{\partial y^4} + \rho h \frac{\partial^2 w(x,y,t)}{\partial t^2} = 0$$

$$D_1 = \frac{E_1 h^3}{12(1-\vartheta_{12}\vartheta_{21})},\ D_2 = \frac{E_2 h^3}{12(1-\vartheta_{12}\vartheta_{21})},\ D_{66} = \frac{G_{12} h^3}{12},\ D_3 = D_{12} + 2D_{66},\ D_{12} = \vartheta_{12} D_2 = \vartheta_{21} D_1$$

If $E_1 = E_2 = E$ & $\vartheta_{12} = \vartheta_{21} = \vartheta$,

$$D_1 = D_2 = \frac{Eh^3}{12(1-\vartheta^2)} = D \ \&\ D_3 = D_{12} + 2D_{66} = \frac{\vartheta E h^3}{12(1-\vartheta^2)} + 2\frac{Eh^3}{12*2*(1-\vartheta)} = \frac{Eh^3}{12(1-\vartheta^2)} = D$$

Then Isotropic Homogenouse case

$$D\Delta\Delta w(x,y,t) + \rho h \frac{\partial^2 w(x,y,t)}{\partial t^2} = 0$$

Eigenfunction 3

SOLUTION FOR HOMOGENOUS ISOTROPIC CASE

$$w(x,y,t) = \Phi(x)\psi(y)$$

Then

$$\frac{\partial^4\Phi(x)}{\partial x^4}\psi(y) + 2\frac{\partial^4\Phi(x)\psi(y)}{\partial x^2\partial y^2} + \frac{\partial^4\psi(y)}{\partial y^4}\Phi(x) = 0$$

$$\Rightarrow \ \Phi(x) = Ae^{\alpha x} \,\&\, \psi(y) = Be^{\beta y}$$

Then

$$\alpha^4 + 2\alpha^2\beta^2 + \beta^4 = 0$$

So

$$\alpha^2{}_{1,2} = -\beta^2 \ or \ \beta^2{}_{1,2} = -\alpha^2$$

$$\alpha = i\beta,\ \alpha = i\beta$$
$$\alpha = -i\beta,\ \alpha = -i\beta \Rightarrow \beta = i\alpha, \beta = i\alpha$$

Eigenfunction 3

SOLUTION FOR HOMOGENOUS ISOTROPIC CASE

The general solution

$$
\begin{aligned}
w(x,y,t) &= (C+Dx)e^{\beta i x}\cdot Be^{\beta y} + (E+Fy)e^{\alpha i y}\cdot Ae^{\alpha x} \\
&= (C+Dx)e^{\alpha x}\cdot Be^{i\alpha y} + (E+Fy)e^{\beta y}\cdot Ae^{i\beta x} \\
&= B(C+Dx)(\sinh(\alpha x) + cosh(\alpha x))\cdot(\sin(\alpha y) + cos(\alpha y)) \\
&\quad + A(E+Fy)(\sinh(\beta y) + cosh(\beta y))\cdot(\sin(\beta x) + cos(\beta x)) \\
&= \sin(\alpha y)\,B(C+Dx)(\sinh(\alpha x) + cosh(\alpha x)) \\
&\quad + cos(\alpha y)\,B(C+Dx)(\sinh(\alpha x) + cosh(\alpha x)) \\
&\quad \sin(\beta x)\,A(E+Fy)(\sinh(\beta y) + cosh(\beta y)) \\
&\quad + cos(\beta x)\,A(E+Fy)(\sinh(\beta y) + cosh(\beta y)) \\
&= \sin(\alpha y)\,[B(C+Dx)\sinh(\alpha x) + B(C+Dx)\,cosh(\alpha x)] \\
&\quad + cos(\alpha y)[\,B(C+Dx)\sinh(\alpha x) + B(C+Dx)\,cosh(\alpha x)] \\
&\quad + \sin(\beta x)\,[A(E+Fy)\sinh(\beta y) + A(E+Fy)\,cosh(\beta y)] \\
&\quad + cos(\beta x)\,[A(E+Fy)\sinh(\beta y) + A(E+Fy)\,cosh(\beta y)]
\end{aligned}
$$

Airy Stress Function approach form

Eigenfunction 2

EIGENFUNCTIONS TEST

If $w = Ae^{\alpha x}\ Be^{\beta y}$ is a solution, $w = Axe^{\alpha x}\ Be^{\beta y}$ is also a solution? $w = Aye^{\alpha x}\ Be^{\beta y}$ or $w = Axye^{\alpha x}\ Be^{\beta y}$? $w = Ax^2e^{\alpha x}\ Be^{\beta y}$, $w = Ay^2e^{\alpha x}\ Be^{\beta y}$ etc.

1) $w = Axe^{\alpha x}\ Be^{\beta y}$

$$\frac{\partial^4 w}{\partial x^4} = 4\alpha^3 Ae^{\alpha x}\ Be^{\beta y} + 4\alpha^4 xAe^{\alpha x}\ Be^{\beta y}$$

$$2\frac{\partial^4 w}{\partial x^2 \partial y^2} = 2(2\alpha\beta^2 Ae^{\alpha x}\ Be^{\beta y} + \alpha^2\beta^2 xAe^{\alpha x}\ Be^{\beta y})$$

$$\frac{\partial^4 w}{\partial y^4} = \beta^4 xAe^{\alpha x}\ Be^{\beta y}$$

$$\Delta\Delta w = \lambda w \Rightarrow 4\alpha^3 = -4\alpha\beta^2$$

$$\Rightarrow \alpha^2 = -\beta^2$$

$$\alpha = i\beta, \alpha = i\beta$$

$$\alpha = -i\beta, \alpha = -i\beta$$

Eigenfunction 2

Eigenfunctions test

If $w = Ae^{\alpha x}\, Be^{\beta y}$ is a solution, $w = Axe^{\alpha x}\, Be^{\beta y}$ is also a solution? $w = Ax^2e^{\alpha x}\, Be^{\beta y}$ or $w = Aye^{\alpha x}\, Be^{\beta y}$? $w = Ay^2e^{\alpha x}\, Be^{\beta y}$ $w = Axye^{\alpha x}\, Be^{\beta y}$ etc.

1) $w = Ax^2e^{\alpha x}\, Be^{\beta y}$

$$\frac{\partial^4 w}{\partial x^4} = 12\alpha^3 Ae^{\alpha x}\, Be^{\beta y} + 8\alpha^3 xAe^{\alpha x}\, Be^{\beta y} + \alpha^4 x^2 Ae^{\alpha x}\, Be^{\beta y}$$

$$2\frac{\partial^4 w}{\partial x^2 \partial y^2} = 2(2\beta^2 Ae^{\alpha x}\, Be^{\beta y} + 4\alpha\beta^2 xAe^{\alpha x}\, Be^{\beta y} + \alpha^2\beta^2 x^2 Ae^{\alpha x}\, Be^{\beta y})$$

$$\frac{\partial^4 w}{\partial y^4} = \beta^4 x^2 Ae^{\alpha x}\, Be^{\beta y}$$

$$\Delta\Delta w = \lambda w \Rightarrow \begin{cases} 12\alpha^2 = -4\beta^2 \\ 8\alpha^3 = -8\alpha\beta^2 \end{cases}$$

$$\Rightarrow \begin{cases} 3\alpha^2 = -\beta^2 \\ \alpha^2 = -\beta^2 \end{cases}$$

Eigenfunction 2

EIGENFUNCTIONS TEST

If $w = Ae^{\alpha x}\ Be^{\beta y}$ is a solution, $w = Axe^{\alpha x}\ Be^{\beta y}$ is also a solution? $w = Ax^2e^{\alpha x}\ Be^{\beta y}$ or $w = Aye^{\alpha x}\ Be^{\beta y}$? $w = Ay^2e^{\alpha x}\ Be^{\beta y}$ $w = Axye^{\alpha x}\ Be^{\beta y}$ etc.

1) $w = Ax^3e^{\alpha x}\ Be^{\beta y}$

$$\frac{\partial^4 w}{\partial x^4} = 24\alpha Ae^{\alpha x}\ Be^{\beta y} + 36\alpha^2 xAe^{\alpha x}\ Be^{\beta y} + 12\alpha^3 x^2 Ae^{\alpha x}\ Be^{\beta y} + \alpha^4 x^3 Ae^{\alpha x}\ Be^{\beta y}$$

$$2\frac{\partial^4 w}{\partial x^2 \partial y^2} = 2(6\beta^2 xAe^{\alpha x}\ Be^{\beta y} + 6\alpha\beta^2 x^2 Ae^{\alpha x}\ Be^{\beta y} + \alpha^2\beta^2 x^3 Ae^{\alpha x}\ Be^{\beta y})$$

$$\frac{\partial^4 w}{\partial y^4} = \beta^4 x^3 Ae^{\alpha x}\ Be^{\beta y}$$

$$\Delta\Delta w \neq \lambda w$$

Conclusion:

$Ae^{\alpha x}\ Be^{\beta y}$, $Axe^{\alpha x}\ Be^{\beta y}$,$Aye^{\alpha x}\ Be^{\beta y}$,$Ay^2e^{\alpha x}\ Be^{\beta y}$,$Axye^{\alpha x}\ Be^{\beta y}$, $Ax^2e^{\alpha x}\ Be^{\beta y}$ etc. can not form a basis of the plate's eigenfunction.

Eigenfunction 2

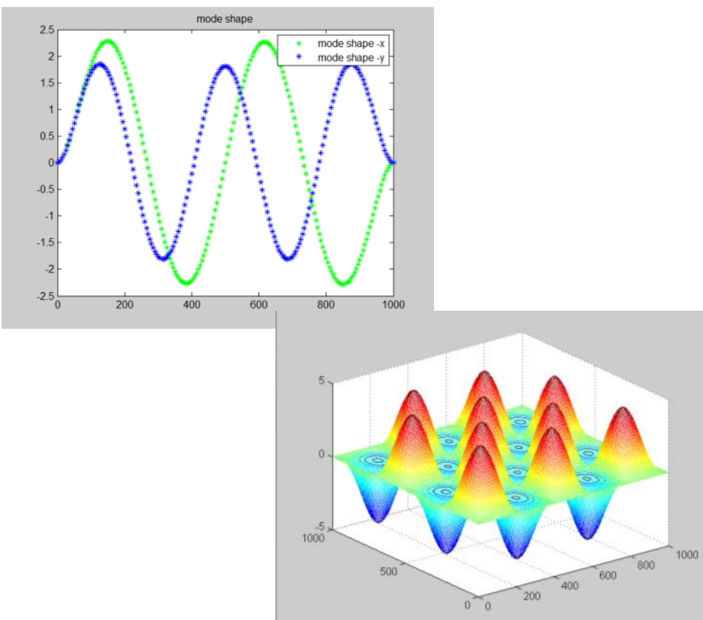
EIGENFUNCTIONS TEST

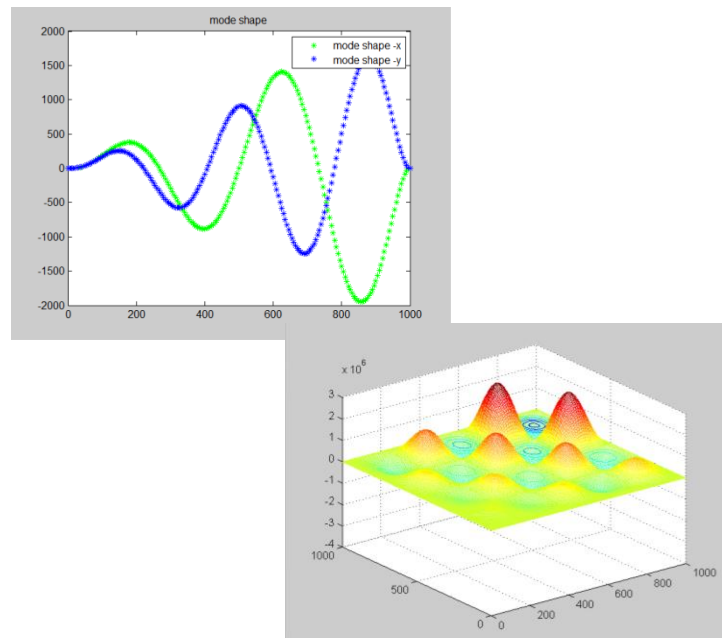
$Ae^{\alpha x}\ Be^{\beta y}$, $Axe^{\alpha x}\ Be^{\beta y}$,$Aye^{\alpha x}\ Be^{\beta y}$,$Ay^2e^{\alpha x}\ Be^{\beta y}$,$Axye^{\alpha x}\ Be^{\beta y}$, $Ax^2e^{\alpha x}\ Be^{\beta y}$ etc. can not form a basis of the plate's eigenfunction.

1) $w = Ae^{\alpha x}\ Be^{\beta y} + Cxe^{\alpha x}\ De^{\beta y}(+Eye^{\alpha x}\ Fe^{\beta y})$

only works for isotropic case

could be a solution of $\Delta\Delta w = \lambda w$, but meanless for plate vibration





Eigenfunction 2

NORMAL EIGENFUNCTIONS BASIS

Normal eigenfunction (C=1):

$$w(x,y) = \Phi(x)\,\psi(y)$$

$$\Phi(x) = -cos\alpha_1 x + (\beta_1/\alpha_1)k_2 sin\alpha_1 x + cosh\beta_1 x - k_2 sinh\beta_1 x$$

$$\psi(y) = -cos\alpha_2 y + (\beta_2/\alpha_2)k_1 sin\alpha_2 y + cosh\beta_2 y - k_1 sinh\beta_2 y$$

Self-joint operator:

$$\langle Lw_1, w_2\rangle = \langle w_1, L^* w_2\rangle \ \ \& L = L^*$$

with $\langle Lu, v\rangle = \iint Lu \cdot vdxdy$

$$\lambda_n - \lambda_m \neq 0 \Rightarrow \langle w_n, w_m\rangle = 0$$

An orthonormal set in a Hilbert space is an orthonormal basis in this Hilbert space.

1. Norm the eigenfunctions
2. The inner product $\langle w_1, w_2\rangle$ form a vector space W?
3. This W is complete (Hilbert space)?

Eigenfunction 2

INNER PRODUCT

$$L = D_1 \frac{\partial^4}{\partial x^4} + 2D_3 \frac{\partial^4}{\partial x^2 \partial y^2} + D_2 \frac{\partial^4}{\partial y^4}$$

Assuming w_1 & w_2 two function satisfy $Lw = \lambda w$ and the boundary conditions $\mathrm{w} = \frac{\partial w}{\partial x} = 0$ for x=0,a and $\mathrm{w} = \frac{\partial w}{\partial y} = 0$ for y=0,b

Strain energy for orthotropic plate:

$$U = \frac{1}{2} \iint D_1 \left(\frac{\partial^2 w}{\partial x^2}\right)^2 + 2D_{12} \frac{\partial^2 w}{\partial x^2} \frac{\partial^2 w}{\partial y^2} + D_2 \left(\frac{\partial^2 w}{\partial y^2}\right)^2 + 4D_{66} \left(\frac{\partial^2 w}{\partial x \partial y}\right)^2 dxdy$$

$$= \frac{1}{2} \iint \left[D_1 \left(\frac{\partial^2 w}{\partial x^2}\right)^2 + 2D_3 \left(\frac{\partial^2 w}{\partial x \partial y}\right)^2 + D_2 \left(\frac{\partial^2 w}{\partial y^2}\right)^2\right] + (D_1 \vartheta_{21} + D_2 \vartheta_{12}) \left[\frac{\partial^2 w}{\partial x^2} \frac{\partial^2 w}{\partial y^2} - \left(\frac{\partial^2 w}{\partial x \partial y}\right)^2\right] dxdy$$

Strain energy for orthotropic plate:

$$U = \frac{1}{2} \iint D \left(\left(\frac{\partial^2 w}{\partial x^2}\right)^2 + 2\vartheta \frac{\partial^2 w}{\partial x^2} \frac{\partial^2 w}{\partial y^2} + \left(\frac{\partial^2 w}{\partial y^2}\right)^2 + 2(1-\vartheta) \left(\frac{\partial^2 w}{\partial x \partial y}\right)^2\right) dxdy$$

$$\langle u, v \rangle = \iint Lu \cdot v dxdy$$

$$\langle w, w \rangle = \iint Lw \cdot w \ dxdy \ \textit{is the bending energy?}$$

Eigenfunction 2

INNER PRODUCT

$$\langle w, w \rangle = \iint Lw \cdot w \, dxdy$$

$$= \iint \left(D_1 \frac{\partial^4 w}{\partial x^4} + 2D_3 \frac{\partial^4 w}{\partial x^2 \partial y^2} + D_2 \frac{\partial^4 w}{\partial y^4} \right) w dxdy$$

$$\int\int \frac{\partial^4 w}{\partial x^4} w \, dxdy = \int \left[\frac{\partial^3 w}{\partial x^3} w \bigg|_{x=0,a} - \int \left(\frac{\partial^3 w}{\partial x^3} \frac{\partial w}{\partial x} \right) dx \right] dy = -\int \left[\int \left(\frac{\partial^3 w}{\partial x^3} \frac{\partial w}{\partial x} \right) dx \right] dy$$

$$= -\int \left[\frac{\partial^2 w}{\partial x^2} \frac{\partial w}{\partial x} \bigg|_{x=0,a} - \int \left(\frac{\partial^2 w}{\partial x^2} \frac{\partial^2 w}{\partial x^2} \right) dx \right] dy = \int \left[\int \left(\frac{\partial^2 w}{\partial x^2} \frac{\partial^2 w}{\partial x^2} \right) dx \right] dy$$

$$\int\int \frac{\partial^4 w}{\partial y^4} w \, dxdy = \int \left[\frac{\partial^3 w}{\partial y^3} w \bigg|_{y=0,b} - \int \left(\frac{\partial^3 w}{\partial y^3} \frac{\partial w}{\partial y} \right) dx \right] dy = -\int \left[\int \left(\frac{\partial^3 w}{\partial y^3} \frac{\partial w}{\partial y} \right) dx \right] dy$$

$$= -\int \left[\frac{\partial^2 w}{\partial y^2} \frac{\partial w}{\partial y} \bigg|_{y=0,b} - \int \left(\frac{\partial^2 w}{\partial y^2} \frac{\partial^2 w}{\partial y^2} \right) dx \right] dy = \int \left[\int \left(\frac{\partial^2 w}{\partial y^2} \frac{\partial^2 w}{\partial y^2} \right) dx \right] dy$$

Eigenfunction 2

INNER PRODUCT

$$\int\int \frac{\partial^4 w}{\partial x^2 \partial y^2} w\, dxdy = \int \left[\frac{\partial^3 w}{\partial x \partial y^2} w \bigg|_{x=0,a} - \int \left(\frac{\partial^3 w}{\partial x \partial y^2} \frac{\partial w}{\partial x} \right) dx \right] dy = -\int \left[\int \left(\frac{\partial^3 w}{\partial x \partial y^2} \frac{\partial w}{\partial x} \right) dx \right] dy$$

$$= -\int \left[\frac{\partial^2 w}{\partial x \partial y} \frac{\partial w}{\partial x} \bigg|_{y=0,b} - \int \left(\frac{\partial^2 w}{\partial x \partial y} \frac{\partial^2 w}{\partial x \partial y} \right) dy \right] dx = \int \int \frac{\partial^2 w}{\partial x \partial y} \frac{\partial^2 w}{\partial x \partial y} dxdy$$

So, for clamped orthotropic plate:

$$U = \langle w, w \rangle = \iint Lw \cdot w\, dxdy$$

Eigenfunction 2

VECTOR SPACE

$\langle u, v\rangle = \iint Lu \cdot v dxdy$ with the eigenfunctions $\Phi_m(x)\psi_n(y)$ (orthogonal set)form a vector space?

Three elements: $w_1 = \sum_m \sum_n A_{mn}\Phi_m\psi_n$, $w_2 = \sum_m \sum_n B_{mn}\Phi_m\psi_n$, $w_3 = \sum_m \sum_n C_{mn}\Phi_m\psi_n$ then

$$\langle w_1, w_2\rangle = \iint Lw_1 \cdot w_2 dxdy = \iint w_1 \cdot Lw_2 dxdy = \langle w_2, w_1\rangle$$

$$\langle \alpha w_1, w_2\rangle = \iint L\alpha w_1 \cdot w_2 dxdy = \alpha \iint w_1 \cdot Lw_2 dxdy = \alpha\langle w_1, w_2\rangle$$

$$\langle w_1 + w_2, w_3\rangle = \iint L(w_1 + w_2) \cdot w_3 dxdy = \iint (Lw_1 + Lw_2) \cdot w_3 dxdy = \iint Lw_1 \cdot w_3 dxdy + \iint Lw_2 \cdot w_3 dxdy$$
$$= \langle w_1, w_3\rangle + \langle w_2, w_3\rangle$$

$$\langle w_1, w_1\rangle = \iint Lw_1 \cdot w_1 dxdy = \iint \lambda_1 w_1 \cdot w_1 dxdy = \iint \lambda_1 ||w_1||^2 dxdy \geq 0$$

$$\langle w_1, w_1\rangle = 0 \Rightarrow w_1 = 0$$

Eigenfunction 2

COMPLETENESS OF THE VECTOR SPACE

A normed vector space V is complete if every Cauchy sequence converges.

Cauchy sequence: for a given $\varepsilon > 0$sequence $w = \sum_m \sum_n A_{mn} \Phi_m \psi_n$ there exist M,N such as $\mathrm{m}, m' > M$ & $\mathrm{n}, n' > N$, $\left|\sum_{m'} \sum_{n'} A_{m'n'} \Phi_{m'} \psi_{n'} - \sum_m \sum_n A_{mn} \Phi_m \psi_n\right| < \varepsilon$.