

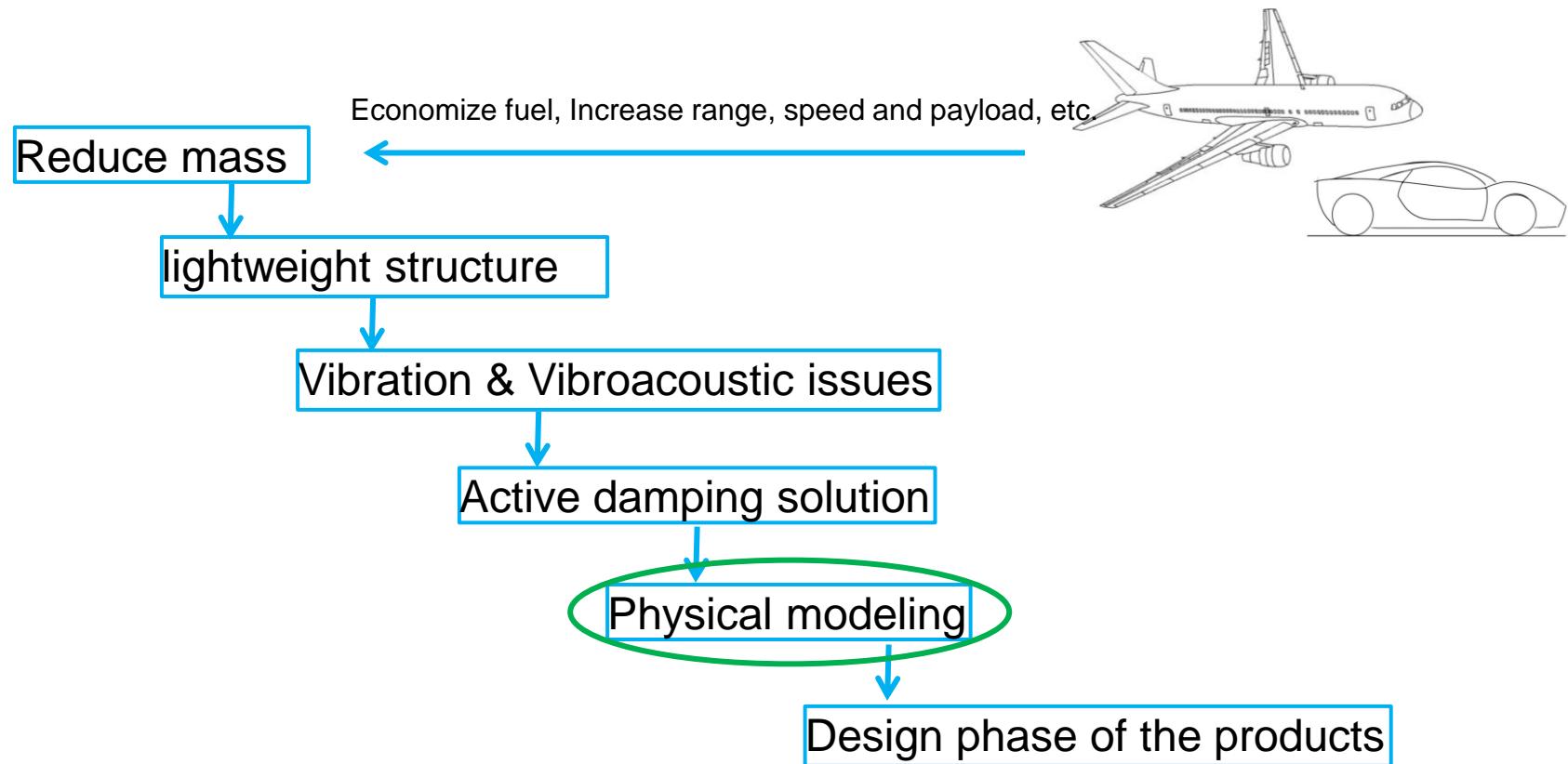
PhD content presentation

Smart structures modelling and model order reduction techniques for feasible distributed active damping solutions

ZhongZhe DONG

Overview of ARRAYCON

MOTIVATION AND RESEARCH DIRECTION:



Overview of ARRAYCON

MOTIVATION AND RESEARCH DIRECTION:

- Assuming the actuators do not affect on the mode shapes:
Piezoelectric patches extract energy from structure => Amplitude attenuation of each mode
 Analytical solution!
- FEM modeling: modal analysis
- Piezoelectric materials study
 1. Complexity of the actuators/sensors
 2. Non linearity of piezoelectric effect (Hysteresis, creep)

Analytical solution

EQUATION OF MOTION:

$$D_1 \frac{\partial^4 w(x, y, t)}{\partial x^4} + 2D_3 \frac{\partial^4 w(x, y, t)}{\partial x^2 \partial y^2} + D_2 \frac{\partial^4 w(x, y, t)}{\partial y^4} + \rho h \frac{\partial^2 w(x, y, t)}{\partial t^2} = 0$$

$$w(x, y, t) = \Phi(x)\psi(y)\exp(J\omega t)$$

Eigen fonction:

$$w_i = \Phi(x)\psi(y)$$

$$\Phi(x) = -\cos\alpha_1 x + (\beta_1/\alpha_1)k_2 \sin\alpha_1 x + \cosh\beta_1 x - k_2 \sinh\beta_1 x$$

$$\psi(y) = -\cos\alpha_2 y + (\beta_2/\alpha_2)k_1 \sin\alpha_2 y + \cosh\beta_2 y - k_1 \sinh\beta_2 y$$

Eigen value:

$$\Omega^2 = \sqrt{\frac{D_1}{4} \left((\alpha_1^2 + \beta_1^2)^2 + \frac{D_1 D_2 - D_3^2}{D_3^2} (\alpha_1^2 - \beta_1^2)^2 \right)}$$

Ref: Y.F. Xing & B. Liu: New exact solutions for free vibrations of thin orthotropic rectangular plates 2008

Analytical solution

OPERATOR DEFINITION:

$$D_1 \frac{\partial^4 w(x, y, t)}{\partial x^4} + 2D_3 \frac{\partial^4 w(x, y, t)}{\partial x^2 \partial y^2} + D_2 \frac{\partial^4 w(x, y, t)}{\partial y^4} + \rho h \frac{\partial^2 w(x, y, t)}{\partial t^2} = 0$$

$$L = D_1 \frac{\partial^4}{\partial x^4} + 2D_3 \frac{\partial^4}{\partial x^2 \partial y^2} + D_2 \frac{\partial^4}{\partial y^4} \Rightarrow Lw + \rho h \frac{\partial^2 w(x, y, t)}{\partial t^2} = 0$$

1. Modes' orthogonality : proved by self-adjoint operator method

$$W_{(x,y,t)} = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \phi_m(x) \psi_n(y) C_{mn} e^{j\Omega_{mn}t}$$

2. Bending energy of a clamped/simple supported plate:

$$U = \frac{1}{2} \iint \left[D_1 \left(\frac{\partial^2 w}{\partial x^2} \right)^2 + 2D_3 \left(\frac{\partial^2 w}{\partial x \partial y} \right)^2 + D_2 \left(\frac{\partial^2 w}{\partial y^2} \right)^2 \right] + (D_1 \vartheta_{21} + D_2 \vartheta_{12}) \left[\frac{\partial^2 w}{\partial x^2} \frac{\partial^2 w}{\partial y^2} - \left(\frac{\partial^2 w}{\partial x \partial y} \right)^2 \right] dxdy$$

$$U = \langle w, w \rangle = \frac{1}{2} \iint Lw \cdot w dxdy$$

Analytical solution

GREEN FUNCTION OF A SINGLE CLAMPED PLATE

$$W_{(x,y,t)} = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \phi_m(x) \psi_n(y) C_{mn} e^{J\Omega_{mn} t}$$

$$\dot{W}_{(\bar{x},\bar{y},t)} = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} J\phi_m(\bar{x}) \psi_n(\bar{y}) C_{mn} e^{J\Omega_{mn} t}$$

$$\dot{W}_{(\bar{x},\bar{y},0)} = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} J\Omega_{mn} \phi_m(x) \psi_n(y) C_{mn} \Rightarrow C_{mn} = \frac{1}{J\Omega_{mn}} \iint \dot{W}_{(\bar{x},\bar{y},t=0)} \phi_m(\bar{x}) \psi_n(\bar{y}) d\bar{x} d\bar{y}$$

$$W_{(x,y,t)} = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \phi_m(x) \psi_n(y) e^{J\Omega_{mn}(t)} \frac{1}{J\Omega_{mn}} \iint \dot{W}_{(\bar{x},\bar{y},0)} \phi_m(\bar{x}) \psi_n(\bar{y}) d\bar{x} d\bar{y}$$

$$W_{(x,y,t,\tau)} = \int \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \phi_m(x) \psi_n(y) e^{J\Omega_{mn}(t-\tau)} \frac{1}{J\Omega_{mn}} \iint f_{(\bar{x},\bar{y},\tau)} \phi_m(\bar{x}) \psi_n(\bar{y}) d\bar{x} d\bar{y} d\tau$$

Analytical solution

Green function of the plate coupled with extra patches

Galerkin method for a single plate:

$$\iint C_i \sum_i [(LW_i - \rho h \Omega^2 W_i)] W_j dx dy = 0$$

The natural frequencies of the plate coupled with extra patches:

$$\Omega_j = \frac{\iint (LW_j) W_j dx dy}{\iint \rho h \Omega^2 W_j W_j dx dy} = \frac{\text{Strain energy}}{\text{Inertia energy}}$$

Weight of each mode:

$$C_j = \frac{1}{J\Omega_j} \left[\iint \dot{W}_{(\bar{x}, \bar{y}, t=0)} W_j(\bar{x}, \bar{y}) d\bar{x} d\bar{y} \right]$$

Finally,

$$W_{(x,y,t)} = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \phi_m(x) \psi_n(y) C_{mn} e^{j\Omega_{mn} t} = \sum_{i=1}^{\infty} W_i C_i e^{j\Omega_i t}$$

Analytical solution

Green function of the plate coupled with extra patches

Single plate system

$$\iint C_i \sum_i [(LW_i - \rho h \Omega^2 W_i)] W_j dx dy + C_i \sum_k \sum_i \iint (L' W_i) W_j d\hat{x} d\hat{y} - C_i \sum_k \rho_k h_k \Omega_j^2 \sum_i \iint W_i W_j d\hat{x} d\hat{y} = 0$$

Strain energy of extra patches

Inertia energy of extra patches

$$\iint \sum_i [(LW_i - \rho h \Omega^2 W_i)] W_j dx dy + \sum_k \sum_i \iint (L' W_i) W_j d\hat{x} d\hat{y} - \sum_k \rho_k h_k \Omega_j^2 \sum_i \iint W_i W_j d\hat{x} d\hat{y} = 0$$

The natural frequencies of the plate coupled with extra patches:

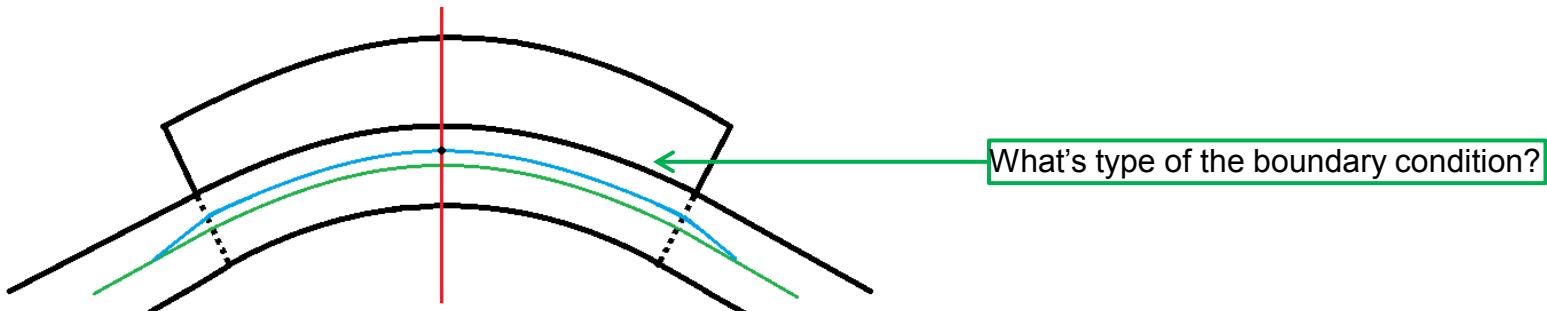
$$\Omega_j^2 = \frac{\iint (LW_j) W_j dx dy + \sum_k \sum_i \iint (L' W_i) W_j d\hat{x} d\hat{y}}{\iint \rho h W_j W_j dx dy + \sum_k \rho_k h_k \sum_i \iint W_i W_j d\hat{x} d\hat{y}}$$

Weight of each mode:

$$C_j = \frac{1}{J\Omega_j} \left[\iint \dot{W}_{(\bar{x}, \bar{y}, t=0)} W_j(\bar{x}, \bar{y}) d\bar{x} d\bar{y} \pm \sum_k \iint \dot{W}_{(\bar{x}, \bar{y}, t=0)} \widehat{W}_j d\hat{x} d\hat{y} \right]$$

Analytical solution

Green function of the plate coupled with extra patches: Problems working on



1. Bending energy of the patch $U = \langle w, w \rangle = \frac{1}{2} \iint Lw \cdot w d\hat{x}d\hat{y}$?
2. The modes of the plate are not locally orthogonal: Physical meaning of $\sum_i \iint W_i W_j d\hat{x}d\hat{y}$?

$$\sum_k \sum_{i=1,2,3} \iint (L'W_i)W_j d\hat{x}d\hat{y} = \sum_k \iint \begin{bmatrix} L'w_1w_1 + L'w_1w_2 + L'w_1w_3 \\ L'w_2w_1 + L'w_2w_2 + L'w_2w_3 \\ L'w_3w_1 + L'w_3w_2 + L'w_3w_3 \end{bmatrix} d\hat{x}d\hat{y}$$

Analytical solution

Green function of the plate coupled with extra patches

By assuming that the modes are independent :

The natural frequencies of the plate coupled with extra patches:

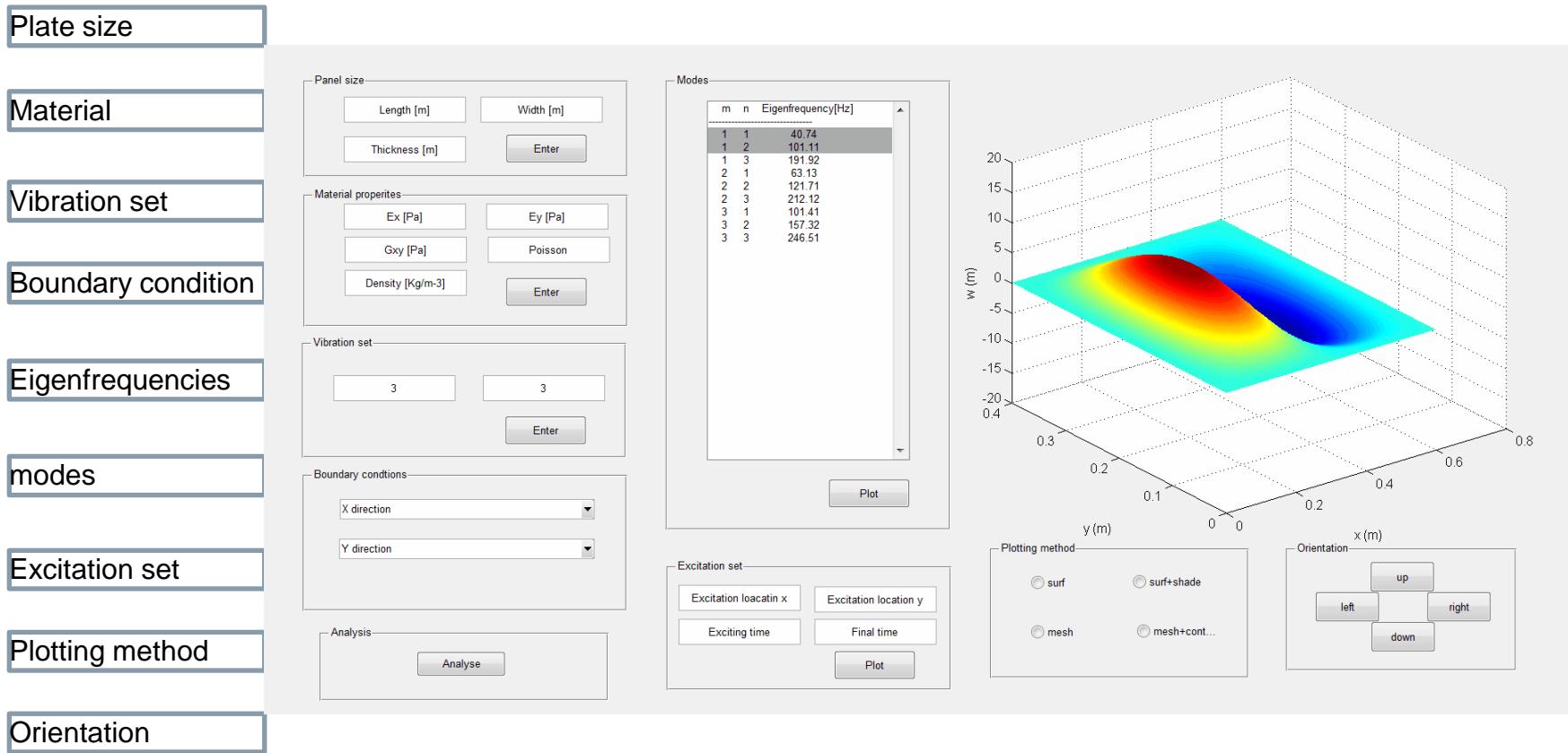
$$\Omega_j = \frac{\iint (LW_j)W_j dx dy + \sum_k \iint (L'W_j)W_j d\hat{x} d\hat{y}}{\iint \rho h W_j W_j dx dy + \sum_k \rho_k h_k \iint W_j W_j d\hat{x} d\hat{y}}$$

Weight of each mode:

$$C_j = \frac{1}{J\Omega_j} \left[\iint \dot{W}_{(\bar{x}, \bar{y}, t=0)} W_j(\bar{x}, \bar{y}) d\bar{x} d\bar{y} \pm \sum_k \iint \dot{W}_{(\bar{x}, \bar{y}, t=0)} \widehat{W}_j d\hat{x} d\hat{y} \right]$$

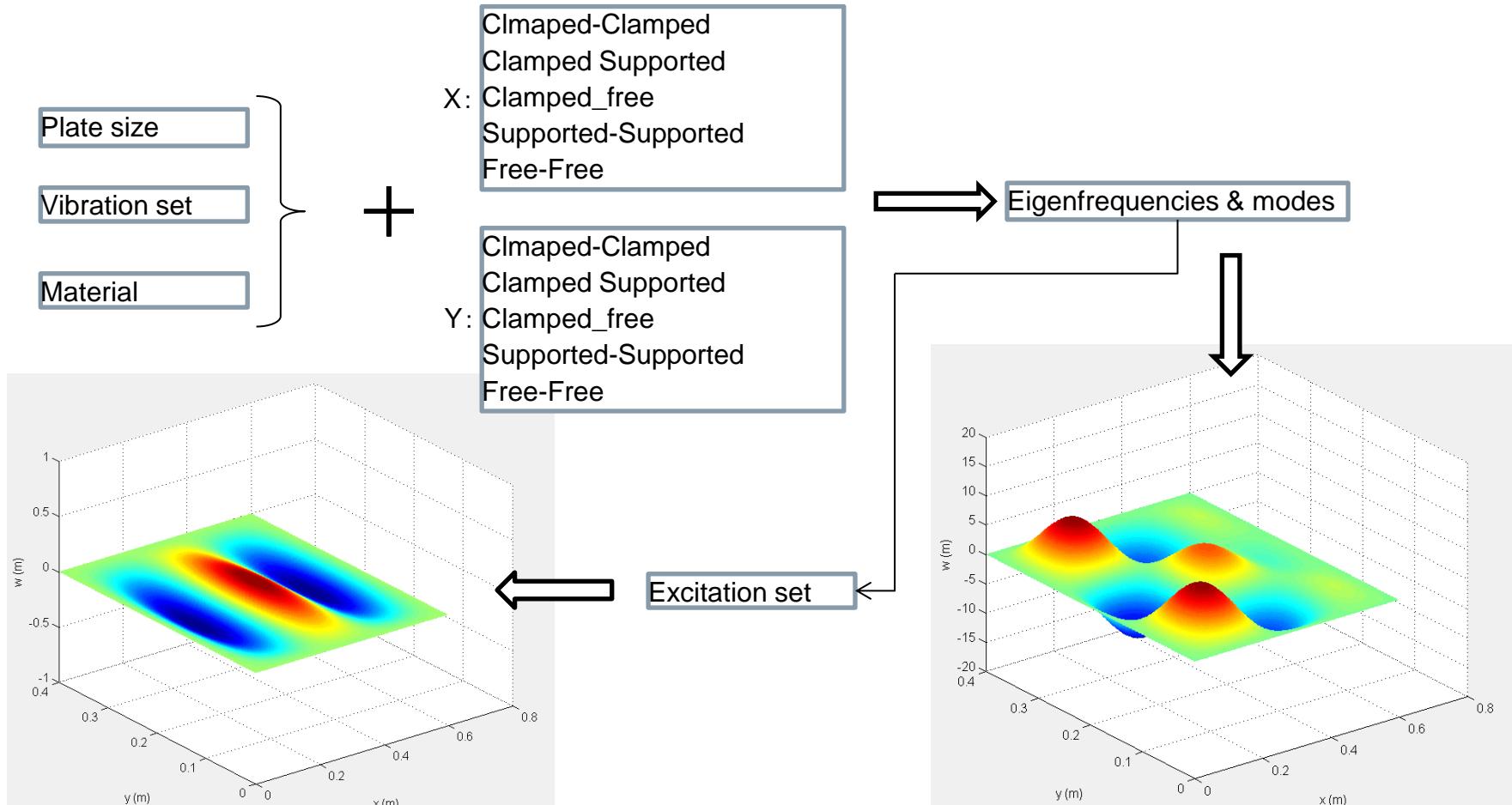
GUI tool

Interface



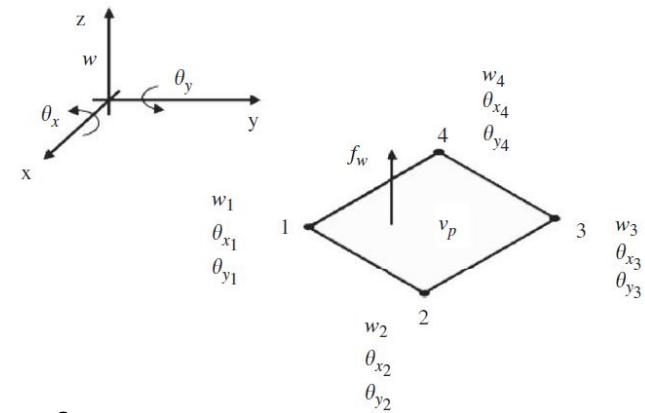
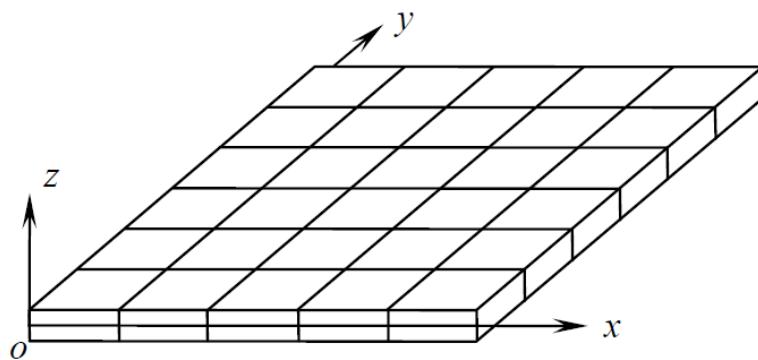
GUI tool

Interface



FEM modeling

Matlab FEM modeling: Kirchhoff plate, Quard element:



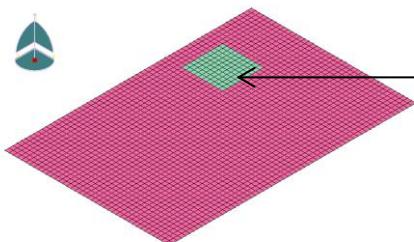
$$\mathbf{m} = \rho_s \int_{V_s} \mathbf{B}_\eta^T \mathbf{Z}^T \mathbf{Z} \mathbf{B}_\eta dV_s + \rho_p \int_{V_p} \mathbf{B}_\eta^T \mathbf{Z}^T \mathbf{Z} \mathbf{B}_\eta dV_p$$

$$\mathbf{k} = \int_{V_s} (\mathbf{z} - \boldsymbol{\delta})^2 \mathbf{B}_k^T \mathbf{c}_s \mathbf{B}_\eta dV_s + \int_{V_p} (\mathbf{z} - \boldsymbol{\delta})^2 \mathbf{B}_k^T \mathbf{c}_p \mathbf{B}_k dV_p$$

$$\mathbf{M}\ddot{\mathbf{u}} + (\mathbf{K} + \mathbf{K}_{ind})\mathbf{u} = \mathbf{0}$$

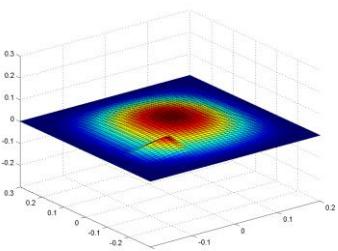
FEM modeling

Mode shape of a clamped Al plate with one extra patch modeling by Matlab

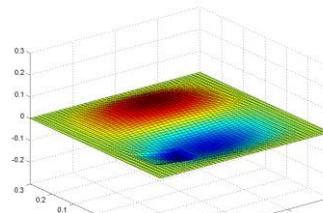


extra patch

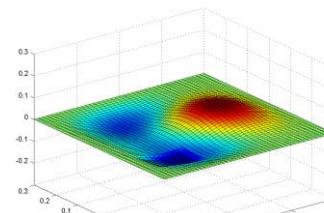
Al plate 600mmx400mmx1mm, extra patch thickness 0.25mm



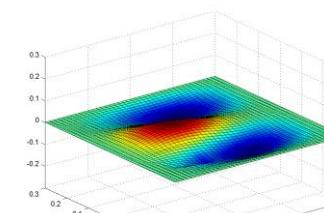
41.3Hz



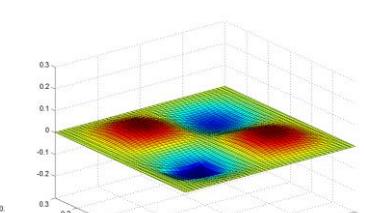
63.4Hz



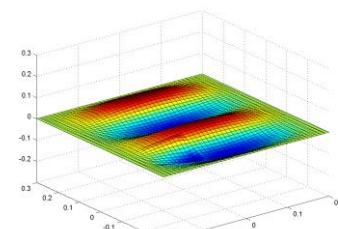
100.8Hz



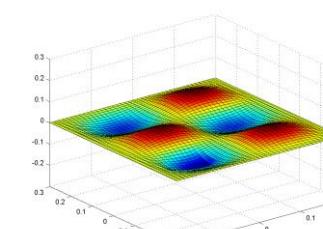
102.5Hz



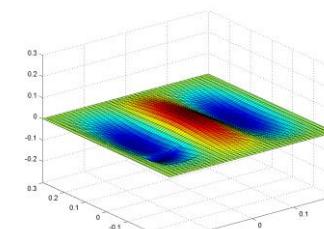
121.8Hz



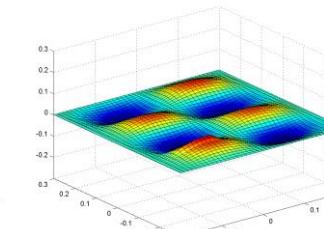
154.8Hz



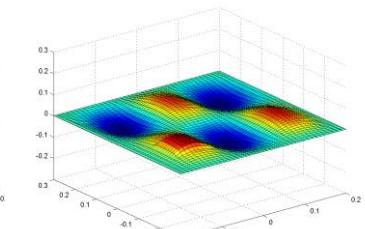
158.0Hz



192.7Hz



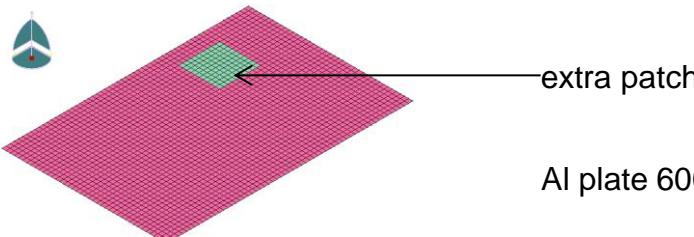
207.7Hz



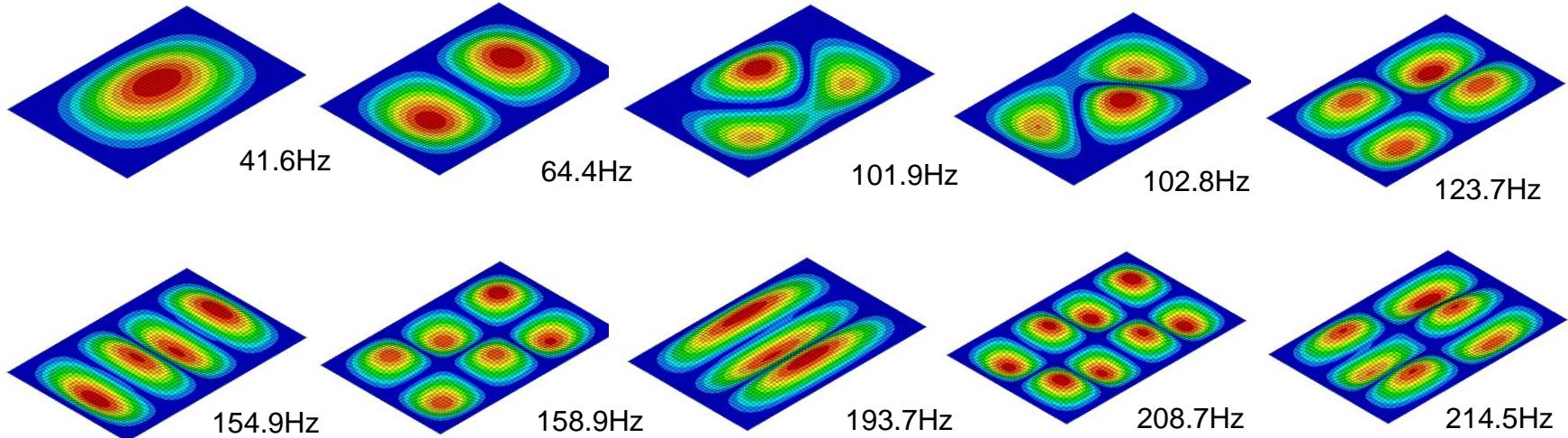
213.6Hz

FEM modeling

Mode shape of a clamped Al plate with one extra patch modeling by LMS virtual lab and NX Nastran



Al plate 600mmx400mmx1mm, extra patch thickness 0.25mm



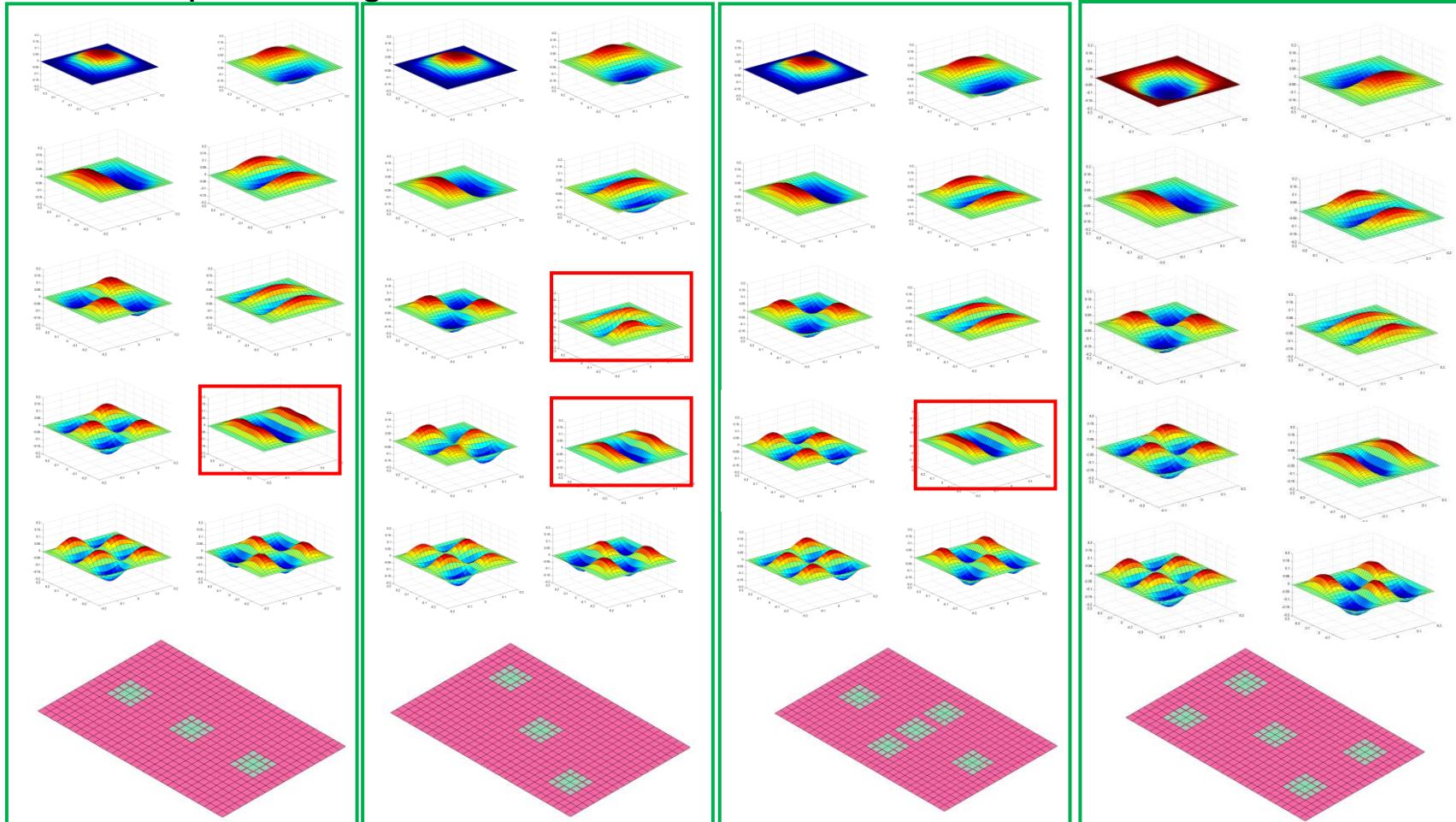
FEM modeling

Eigen frequencies validation:

mode	Plate with one piezo patch by Virtual lab & NX Nastran [Hz]	Plate with one piezo patch by Matlab [Hz]
1	41.6	41.4
2	64.4	63.5
3	101.9	100.8
4	102.8	102.5
5	123.7	121.8
6	154.9	154.8
7	158.9	158.0
8	193.7	192.7
9	208.7	207.7
10	214.5	213.6

FEM modeling

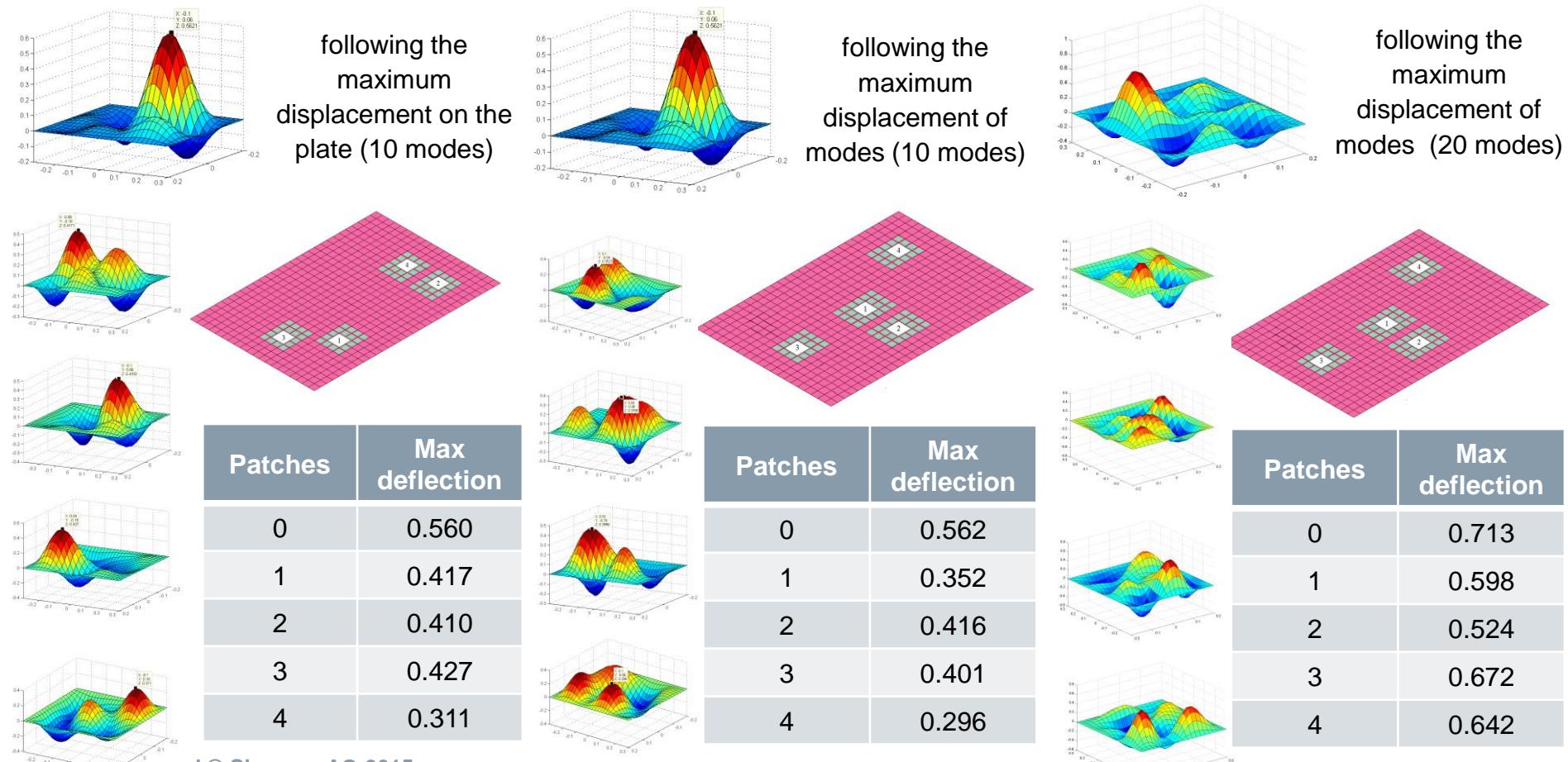
Patches positioning



FEM modeling

Patches positioning

The weight of the modes are not considered here.



Eigen frequencies validation

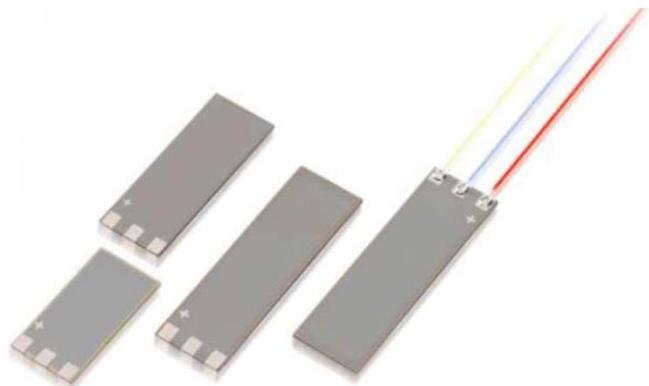
Al plate 600mmx400mmx1mm

mode	Single plate by Virtual lab & NX Nastran [Hz]	Single plate FEM by Matlab [Hz]	Single plate Analytical solution [Hz]	Single plate Galerkin method [Hz]
1	41.5	41.3	40.7	41.4
2	63.9	63.7	63.1	64.0
3	101.7	101.2	101.1	101.4
4	101.9	101.6	101.4	102.0
5	121.9	121.8	121.7	122.3
6	154.5	154.0	--	--
7	156.7	156.9	157.3	158.1
8	192.8	191.8	191.9	192.0
9	206.1	206.6	--	--
10	211.8	211.4	212.1	212.5

Piezo project

PL112 – PL140

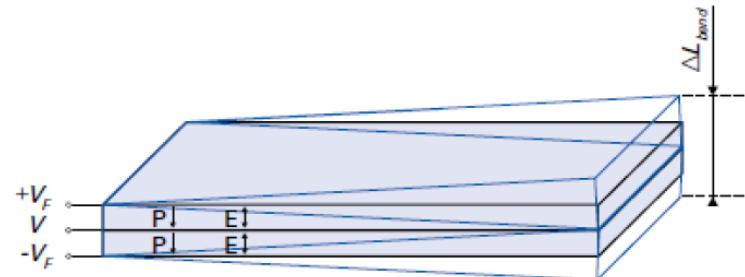
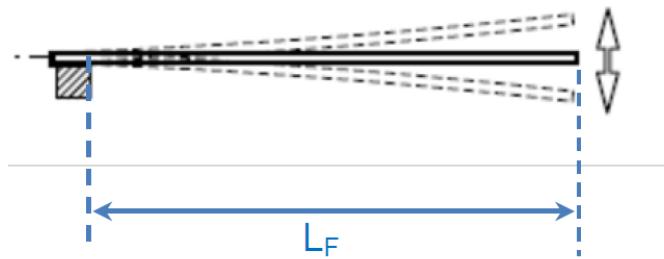
Rectangular PICMA® Bender Piezo Actuators



1. Complexity of the piezo actuators/sensors
 - Multiple layers of piezo crystals, electrodes
 - Boundary condition of the actuators
 - Simplified modeling in numerical simulation
2. Effective properties of the actuators/sensors
 - Could be used in the project (experiments)

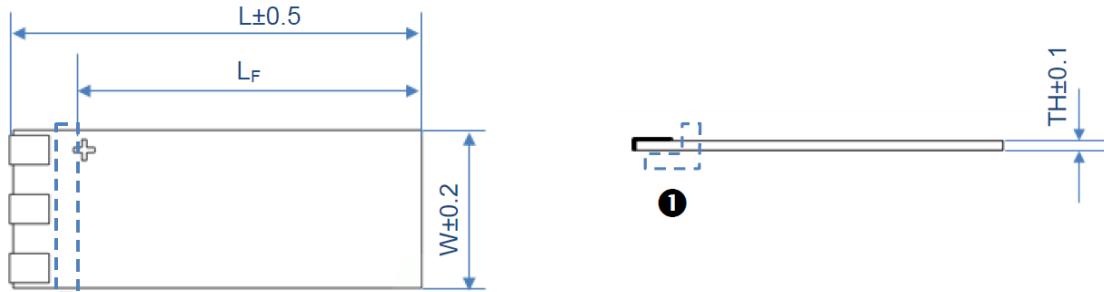
Piezo project: numerical modeling + experiment

Piezo project



1. Detail model in SAMCEF (layers and electrodes)
2. Benchmark against 1D (Beam model)
3. Execute experimental campaign (single clamped-free actuator)
 - With mechanical excitation
 - With electrical excitation
4. Comparison between simulation and experiments

Piezo project



	PL112.10 PL112.11*	PL122.10 PL122.11*	PL127.10 PL127.11*	PL128.10 PL128.11*	PL140.10 PL140.11*	Unit	Toler- ance
Operating voltage**	0 to 60 or -30 to 30	V					
Displacement	±80	±250	±450	±450	±1000	µm	±20%
Free length L_F	12	22	27	28	40	mm	
Dimensions L × W × TH	18.0 × 9.6 × 0.65	25.0 × 9.6 × 0.65	31.0 × 9.6 × 0.65	36.0 × 6.3 × 0.75	45.0 × 11.0 × 0.6	mm ³	(see. abv.)
Blocking force	±2.0	±1.1	±1.0	±0.5	±0.5	N	±20%
Electrical capacitance***	2 * 1.1	2 * 2.4	2 * 3.4	2 * 1.2	2 * 4.0	µF	±20%
Resonant frequency****	2000	660	380	360	160	Hz	±20%
Operating temperature	-20 to 150	-20 to 85	-20 to 85	-20 to 150	-20 to 85	°C	
Piezo ceramic type	PIC252	PIC 251	PIC 251	PIC252	PIC 251		

Piezo project

Piezo Ceramic material: PZT 5H properties

Compliance

16.5 -4.78 -8.45 0 0 0

-4.78 16.5 -8.45 0 0 0

$$\mathbf{s}_E = \begin{matrix} -8.45 & -8.45 & 20.7 & 0 & 0 & 0 \\ 0 & 0 & 0 & 43.5 & 0 & 0 \end{matrix} * 10^{-12} \frac{\text{m}^2}{\text{N}}$$

0 0 0 0 43.5 0

0 0 0 0 0 42.6

Relative Permittivity

3130 0 0

0 3130 0

0 0 3400

$$\frac{\epsilon_T}{\epsilon_0} = \begin{matrix} 3130 & 0 & 0 \\ 0 & 3130 & 0 \end{matrix}, \quad \epsilon_0 = 8.854 * 10^{-12} \frac{\text{F}}{\text{m}}$$

Piezoelectric Coupling

0 0 0 0 741 0

$$\mathbf{d} = \begin{matrix} 0 & 0 & 0 & 741 & 0 & 0 \\ 0 & 0 & 0 & 741 & 0 & 0 \end{matrix} * 10^{-12} \frac{\text{C}}{\text{N}}$$

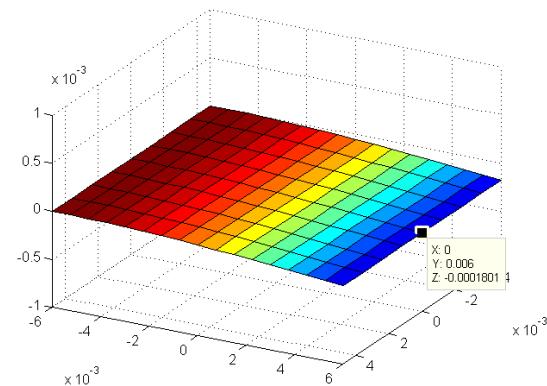
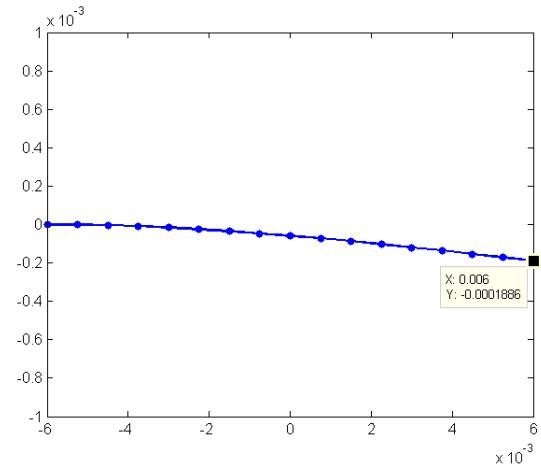
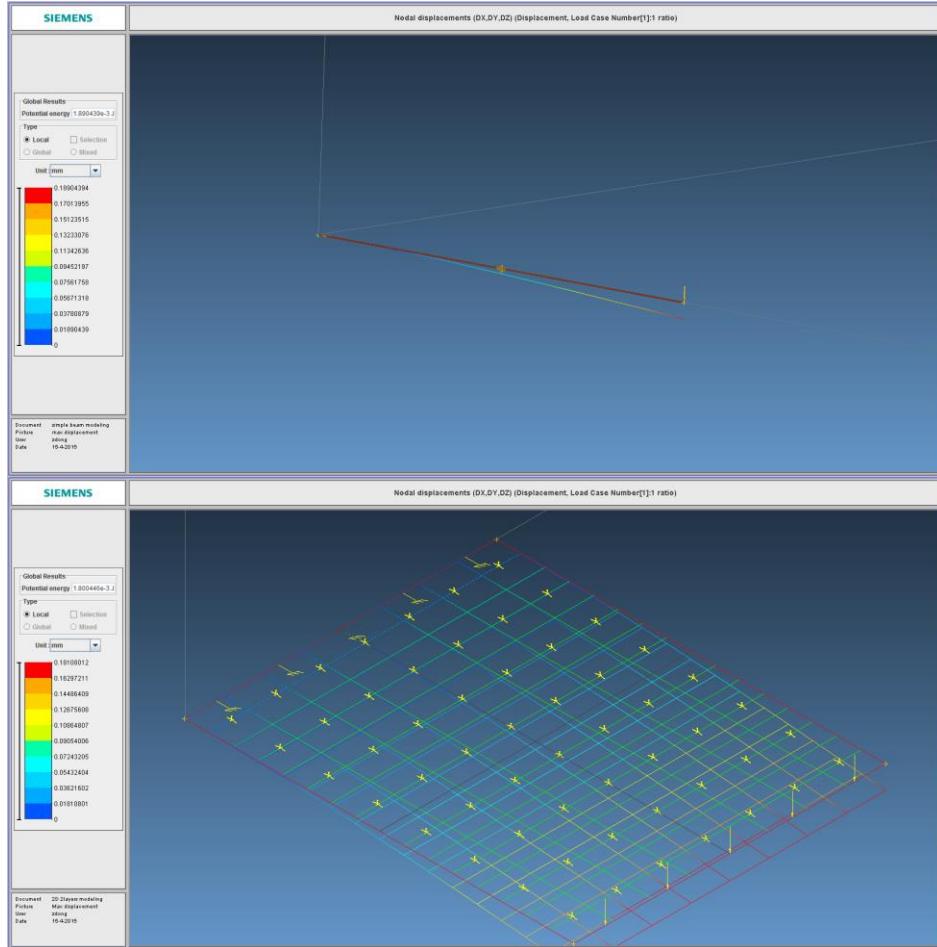
-274 -274 593 0 0 0

Rectangular section Beam/Plate	
Length	12 [mm]
width	9.6 [mm]
Thickness	0.65 [mm]
E	139e9 [Pa]
Poisson ratio	0.29
density	7500 [kg/m^3]

Source: www.efunda.com/materials/piezo/material_data

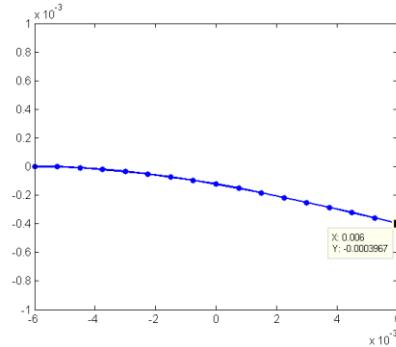
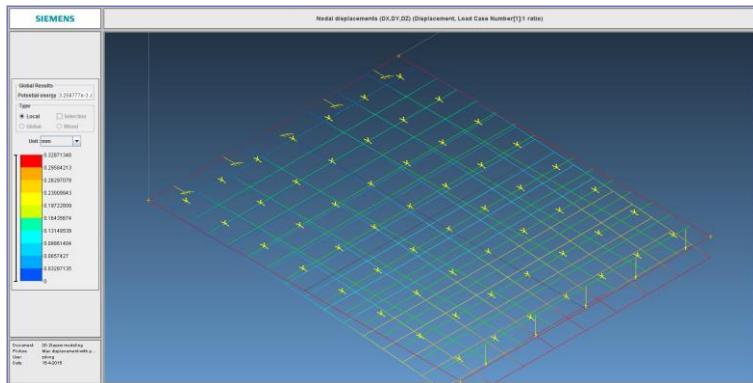
Piezo project

Matlab static modeling validation: Displacement in z direction: F=10N

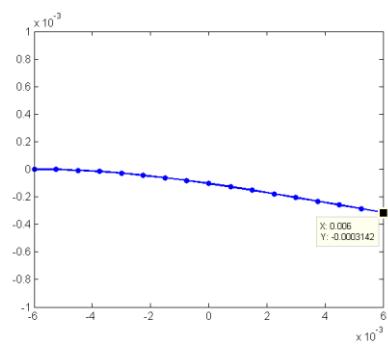


Piezo project

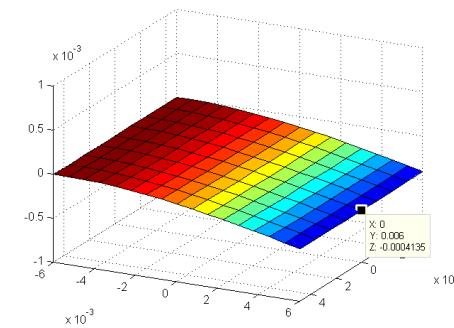
Matlab piezoelectric modeling validation: Displacement in z direction: F=10N



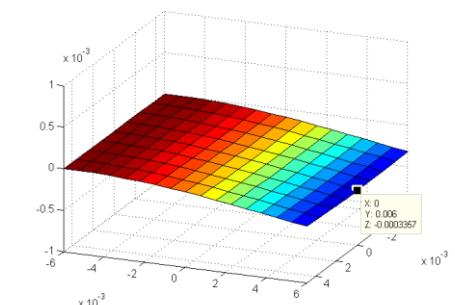
Bending without
piezo effect
0.3967mm



Bending with
piezo effect
0.3142mm



Bending without
piezo effect
0.4135mm



Bending with
piezo effect
0.3377mm

Piezo project

Piezoelectric modeling validation: Voltage produced by force: $F=10N$

Rectangular section Beam/Plate	
Length	12 [mm]
width	9.6 [mm]
Thickness	0.65 [mm]
E	66e9 [Pa]
Poisson ratio	0.29
density	7500 [kg/m ³]

Modeling	Voltage output
Matlab 1D	227.91V
Matlab 2D	250.85V

Matlab modeling calculate the voltage difference between $X=0mm$ and $X=12mm$

Section [mm]	Collective edof	Individual edofs	
X=0.75	107.29V	250.45V	235.38V
X=1.5		220.36V	
X=2.25		197.37V	187.00V
X=3		176.73V	
X=3.75		158.15V	149.56V
X=4.5		141.03V	
X=5.25		124.96V	117.30V
X=6		109.68V	
X=6.75		94.96V	87.79V
X=7.5		80.64V	
X=8.25		66.57V	59.58V
X=9		52.62V	
X=9.75		38.67V	31.60V
X=10.5		24.59V	
X=11.25		10.26V	2.70V
X=12		-4.59V	
edofs average	107.29V	108.90V	108.86V

Simulation result

Static modeling F=10N

Static modeling	Max displacement [mm]	Error	Error
		Compare with SAMCEF 1D	Compare with SAMCEF 2D
SAMCEF 1D	0.189	--	
SAMCEF 2D	0.181	--	
Matlab 1D	0.189	0%	4.42%
Matlab 2D	0.180	-4.76%	-5.5%

The 1D matlab simulation have a larger error:

The structure is more like a plate than a beam.

Piezoelectric effect static modeling F=10N

Simulation	Maximum displacement [mm]	Maximum displacement [mm]	Error
SAMCEF 2D	0.3814 (0.3981 for 1D)	0.3287	--
Matlab 1D	0.3967	0.3142	-4.41%
Matlab 2D	0.4135	0.3377	2.74%

Voltage bending V=38V

Simulation	Maximum displacement [μm]	Error
SAMCEF 2D	6.507--5.735	--
Matlab 1D	5.327	-7.11%
Matlab 2D	6.291—5.561	-3.32%-- -3.03%

Conclusion

ARRAYCON

- Analytical study: Green function for the whole system (plate + extra patches)
- GUI tool serve to the control validation
- FEM modeling serve to the piezoelectric actuators/sensors coupling study
- Analytical solution and FEM solution validation
- Study of a piezo actuator (Piezo project)

Thanks for your attention!

Orthogonality

PROOF OF SELF-ADJOINT OPERATOR

$$L = D_1 \frac{\partial^4}{\partial x^4} + 2D_3 \frac{\partial^4}{\partial x^2 \partial y^2} + D_2 \frac{\partial^4}{\partial y^4}$$

Assuming w_1 & w_2 two function satisfy $Lw = \lambda w$ and the boundary conditions $w = \frac{\partial w}{\partial x} = 0$ for $x=0,a$ and $w = \frac{\partial w}{\partial y} = 0$ for $y=0,b$

$$\langle Lw_1, w_2 \rangle = \iint Lw_1 \cdot w_2 dx dy$$

$$\langle Lw_1, w_2 \rangle = \langle w_1, L^* w_2 \rangle \Rightarrow L = L^*?$$

$$\begin{aligned} \iint \frac{\partial^4 w_1}{\partial x^4} w_2 dx dy &= \int \left[\frac{\partial^3 w_1}{\partial x^3} w_2 \Big|_{x=0,a} - \int \left(\frac{\partial^3 w_1}{\partial x^3} \frac{\partial w_2}{\partial x} \right) dx \right] dy = - \int \left[\int \left(\frac{\partial^3 w_1}{\partial x^3} \frac{\partial w_2}{\partial x} \right) dx \right] dy \\ &= - \int \left[\frac{\partial^2 w_1}{\partial x^2} \frac{\partial w_2}{\partial x} \Big|_{x=0,a} - \int \left(\frac{\partial^2 w_1}{\partial x^2} \frac{\partial^2 w_2}{\partial x^2} \right) dx \right] dy = \int \left[\int \left(\frac{\partial^2 w_1}{\partial x^2} \frac{\partial^2 w_2}{\partial x^2} \right) dx \right] dy \\ &= \int \left[\frac{\partial w_1}{\partial x} \frac{\partial^2 w_2}{\partial x^2} \Big|_{x=0,a} - \int \left(\frac{\partial w_1}{\partial x} \frac{\partial^3 w_2}{\partial x^3} \right) dx \right] dy = - \int \left[\int \left(\frac{\partial w_1}{\partial x} \frac{\partial^3 w_2}{\partial x^3} \right) dx \right] dy = - \int \left[w_1 \frac{\partial^3 w_2}{\partial x^3} \Big|_{x=0,a} - \int \left(w_1 \frac{\partial^4 w_2}{\partial x^4} \right) dx \right] dy \\ &= \iint w_1 \frac{\partial^4 w_2}{\partial x^4} dx dy \end{aligned}$$

Orthogonality

PROOF OF SELF-ADJOINT OPERATOR

In the same way,

$$\int \int \frac{\partial^4 w_1}{\partial y^4} w_2 dx dy = \int \int w_1 \frac{\partial^4 w_2}{\partial y^4} dx dy$$

we know that $w_1 = w_1^1(x) \cdot w_1^2(y)$ & $w_2 = w_2^1(x) \cdot w_2^2(y)$

$$\frac{\partial^4 w_1}{\partial x^2 \partial y^2} w_2 = \frac{\partial^2 w_1^1}{\partial x^2} \frac{\partial^2 w_1^2}{\partial y^2} w_2^1 w_2^2$$

$$\begin{aligned} \int \int \frac{\partial^2 w_1^1}{\partial x^2} \frac{\partial^2 w_1^2}{\partial y^2} w_2^1 w_2^2 dx dy &= \int \left[\frac{\partial w_1^1}{\partial x} w_2^1 \frac{\partial^2 w_1^2}{\partial y^2} w_2^2 \Big|_{x=0,a} - \int \left(\frac{\partial w_1^1}{\partial x} \frac{\partial w_2^1}{\partial x} \frac{\partial^2 w_1^2}{\partial y^2} w_2^2 \right) dx \right] dy \\ &= - \int \left[\int \left(\frac{\partial w_1^1}{\partial x} \frac{\partial w_2^1}{\partial x} \frac{\partial^2 w_1^2}{\partial y^2} w_2^2 \right) dx \right] dy = - \int \left[w_1^1 \frac{\partial w_2^1}{\partial x} \frac{\partial^2 w_1^2}{\partial y^2} w_2^2 \Big|_{x=0,a} - \int \left(w_1^1 \frac{\partial^2 w_2^1}{\partial x^2} \frac{\partial^2 w_1^2}{\partial y^2} w_2^2 \right) dx \right] dy \\ &= \int \int w_1^1 \frac{\partial^2 w_2^1}{\partial x^2} \frac{\partial^2 w_1^2}{\partial y^2} w_2^2 dx dy = \dots = \int \int w_1^1 w_1^2 \frac{\partial^2 w_2^1}{\partial x^2} \frac{\partial^2 w_2^2}{\partial y^2} dx dy \end{aligned}$$

Orthogonality

PROOF OF SELF-ADJOINT OPERATOR

So

$$\langle Lw_1, w_2 \rangle = \langle w_1, L^*w_2 \rangle \quad \& L = L^*$$

with $\langle Lu, v \rangle = \iint Lu \cdot v dx dy$

Then

$$\langle Lw_n, w_m \rangle - \langle w_n, Lw_m \rangle = 0$$

$$\langle \lambda_n - \lambda_m \rangle \cdot \langle w_n, w_m \rangle = 0$$

$$\lambda_n - \lambda_m \neq 0 \Rightarrow \langle w_n, w_m \rangle = 0$$

Inner product

STRAIN ENERGY

$$L = D_1 \frac{\partial^4}{\partial x^4} + 2D_3 \frac{\partial^4}{\partial x^2 \partial y^2} + D_2 \frac{\partial^4}{\partial y^4}$$

Assuming w_1 & w_2 two function satisfy $Lw = \lambda w$ and the boundary conditions $w = \frac{\partial w}{\partial x} = 0$ for $x=0,a$ and $w = \frac{\partial w}{\partial y} = 0$ for $y=0,b$

Strain energy for orthotropic plate:

$$\begin{aligned} U &= \frac{1}{2} \iint D_1 \left(\frac{\partial^2 w}{\partial x^2} \right)^2 + 2D_{12} \frac{\partial^2 w}{\partial x^2} \frac{\partial^2 w}{\partial y^2} + D_2 \left(\frac{\partial^2 w}{\partial y^2} \right)^2 + 4D_{66} \left(\frac{\partial^2 w}{\partial x \partial y} \right)^2 dxdy \\ &= \frac{1}{2} \iint \left[D_1 \left(\frac{\partial^2 w}{\partial x^2} \right)^2 + 2D_3 \left(\frac{\partial^2 w}{\partial x \partial y} \right)^2 + D_2 \left(\frac{\partial^2 w}{\partial y^2} \right)^2 \right] + (D_1 \vartheta_{21} + D_2 \vartheta_{12}) \left[\frac{\partial^2 w}{\partial x^2} \frac{\partial^2 w}{\partial y^2} - \left(\frac{\partial^2 w}{\partial x \partial y} \right)^2 \right] dxdy \end{aligned}$$

Strain energy for isotropic plate:

$$\begin{aligned} U &= \frac{1}{2} \iint D \left(\left(\frac{\partial^2 w}{\partial x^2} \right)^2 + 2\vartheta \frac{\partial^2 w}{\partial x^2} \frac{\partial^2 w}{\partial y^2} + \left(\frac{\partial^2 w}{\partial y^2} \right)^2 + 2(1-\vartheta) \left(\frac{\partial^2 w}{\partial x \partial y} \right)^2 \right) dxdy \\ \langle u, v \rangle &= \iint Lu \cdot v dxdy \end{aligned}$$

$\langle w, w \rangle = \iint Lw \cdot w dxdy$ is the bending energy?

Inner product

STRAIN ENERGY

$$\begin{aligned}
 \langle w, w \rangle &= \iint Lw \cdot w \, dx dy \\
 &= \iint \left(D_1 \frac{\partial^4 w}{\partial x^4} + 2D_3 \frac{\partial^4 w}{\partial x^2 \partial y^2} + D_2 \frac{\partial^4 w}{\partial y^4} \right) w \, dx dy \\
 \iint \frac{\partial^4 w}{\partial x^4} w \, dx dy &= \int \left[\frac{\partial^3 w}{\partial x^3} w \Big|_{x=0,a} - \int \left(\frac{\partial^3 w}{\partial x^3} \frac{\partial w}{\partial x} \right) dx \right] dy = - \int \left[\int \left(\frac{\partial^3 w}{\partial x^3} \frac{\partial w}{\partial x} \right) dx \right] dy \\
 &= - \int \left[\frac{\partial^2 w}{\partial x^2} \frac{\partial w}{\partial x} \Big|_{x=0,a} - \int \left(\frac{\partial^2 w}{\partial x^2} \frac{\partial^2 w}{\partial x^2} \right) dx \right] dy = \int \left[\int \left(\frac{\partial^2 w}{\partial x^2} \frac{\partial^2 w}{\partial x^2} \right) dx \right] dy \\
 \iint \frac{\partial^4 w}{\partial y^4} w \, dx dy &= \int \left[\frac{\partial^3 w}{\partial y^3} w \Big|_{y=0,b} - \int \left(\frac{\partial^3 w}{\partial y^3} \frac{\partial w}{\partial y} \right) dx \right] dy = - \int \left[\int \left(\frac{\partial^3 w}{\partial y^3} \frac{\partial w}{\partial y} \right) dx \right] dy \\
 &= - \int \left[\frac{\partial^2 w}{\partial y^2} \frac{\partial w}{\partial y} \Big|_{y=0,b} - \int \left(\frac{\partial^2 w}{\partial y^2} \frac{\partial^2 w}{\partial y^2} \right) dx \right] dy = \int \left[\int \left(\frac{\partial^2 w}{\partial y^2} \frac{\partial^2 w}{\partial y^2} \right) dx \right] dy
 \end{aligned}$$

Inner product

STRAIN ENERGY

$$\begin{aligned} \int \int \frac{\partial^4 w}{\partial x^2 \partial y^2} w \, dx dy &= \int \left[\frac{\partial^3 w}{\partial x \partial y^2} w \Big|_{x=0,a} - \int \left(\frac{\partial^3 w}{\partial x \partial y^2} \frac{\partial w}{\partial x} \right) dx \right] dy = - \int \left[\int \left(\frac{\partial^3 w}{\partial x \partial y^2} \frac{\partial w}{\partial x} \right) dx \right] dy \\ &= - \int \left[\frac{\partial^2 w}{\partial x \partial y} \frac{\partial w}{\partial x} \Big|_{y=0,b} - \int \left(\frac{\partial^2 w}{\partial x \partial y} \frac{\partial^2 w}{\partial x \partial y} \right) dy \right] dx = \int \int \frac{\partial^2 w}{\partial x \partial y} \frac{\partial^2 w}{\partial x \partial y} \, dx dy \end{aligned}$$

So, for clamped orthotropic plate:

$$U = \langle w, w \rangle = \iint Lw \cdot w \, dx dy$$